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Two compressible immiscible fluids in porous media ☆

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Abstract

We consider a model of flow of two compressible and immiscible phases in a three-dimensional porous media. The equations are obtained by the conservation of the mass of each phase. This model is treated in its general form with the whole nonlinear terms. The only assumption concerns the dependence of densities on a global pressure. We obtain the existence of weak solutions under different kinds of degeneracies of the capillary terms.

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1. Introduction and model

A rigorous mathematical study of the immiscible flow models has been initiated in [1,3,4,6]. In [4], these models are described by using the feature of "global pressure." This approach enables to write all models with one pressure variable and one or several saturations.

The study of two incompressible phases is well known [1,3–6], this is not the case for two compressible phases. The models developed by [4] use the feature of global pressure even if the density of each phase depends on its own pressure. The context is then to assume small capillary pressure so that the densities are assumed to depend on the global pressure.

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We give below the basic model written in variable "global pressure" and saturation.

The equations describing the immiscible displacement of two compressible fluids are given by the following mass conservation of each phase:

$$\phi(x)\partial_t(\rho_i s_i)(t,x) + \operatorname{div}(\rho_i \mathbf{V}_i)(t,x) + \rho_i s_i f_P(t,x) = \rho_i s_i^I f_I(t,x), \quad i = 1,2, \tag{1.1}$$

where ϕ is the porosity of the medium, ρ_i and s_i are respectively the density and the saturation of the ith fluid. The velocity of each fluid $\mathbf{V}_i$ is given by the Darcy law:

$$\mathbf{V}_i(t,x) = -\mathbf{K}(x)\frac{k_i(s_i(t,x))}{\mu_i}\big(\nabla p_i(t,x) - \rho_i(p_i)\mathbf{g}\big), \quad i = 1,2, \tag{1.2}$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, k_i the relative permeability of the ith phase, μ_i the constant i-phase's viscosity and p_i the i-phase's pressure and $\mathbf{g}$ is the gravity term. Here the functions f_I and f_P are respectively the injection and production terms. Note that in Eq. (1.1) the injection term is multiplied by a known saturation s_i^I corresponding to the known injected fluid, whereas the production term is multiplied by the unknown saturation s_i corresponding to the produced fluid. By definition of saturations, one has

$$s_1(t,x) + s_2(t,x) = 1. \tag{1.3}$$

The curvature of the contact surface between the two fluids links the jump of pressure of the two phases to the saturation by the capillary pressure law,

$$p_{12}\big(s_1(t,x)\big) = p_1(t,x) - p_2(t,x). \tag{1.4}$$

With the arbitrary choice of (1.4) (the jump of pressure is a function of s_1), the application $s_1 \mapsto p_{12}(s_1)$ is non-decreasing, ($\frac{dp_{12}}{ds_1}(s_1) \geqslant 0$, for all $s_1 \in [0,1]$).

The two relations (1.3) and (1.4) close the system (1.1). Nevertheless, this system is complex and we prefer to reduce the problem to two unknowns, one saturation s_1 and only one pressure. We then introduce the global pressure. For that, let us denote,

$$\begin{aligned} M_i(s_i) &= k_i(s_i)/\mu_i && i\text{-phase's mobility,}\\ M(s_1) &= M_1(s_1) + M_2(1-s_1) && \text{the total mobility,}\\ \mathbf{V} &= \mathbf{V}_1 + \mathbf{V}_2 && \text{the total velocity.} \end{aligned}$$

As in [4,11] we can express the total velocity in terms of p_2 and p_{12}. We have

$$\begin{aligned} \mathbf{V}(t,x) = {} & -\mathbf{K}(x)M(s_1)\left(\nabla p_2(t,x) + \frac{M_1(s_1)}{M(s_1)}\nabla p_{12}(s_1)\right)\\ & + \mathbf{K}(x)\big(M_1(s_1)\rho_1(p_1) + M_2(s_2)\rho_2(p_2)\big)\mathbf{g}, \end{aligned}$$

and defining a function $\tilde{p}(s_1)$ such that $\frac{d\tilde{p}}{ds}(s_1) = \frac{M_1(s_1)}{M(s_1)}\frac{dp_{12}}{ds}(s_1)$, the global pressure is then $p = p_2 + \tilde{p}$, as in [4],

$$\mathbf{V}(t,x) = -\mathbf{K}(x)M(s_1)\nabla p(t,x) + \mathbf{K}(x)\big(M_1(s_1)\rho_1(p_1) + M_2(s_2)\rho_2(p_2)\big)\mathbf{g}. \tag{1.5}$$

Thus, each phase velocity can be written as

$$\mathbf{V}_i = -\mathbf{K}M_i(s_i)\nabla p - \mathbf{K}\alpha(s_1)\nabla s_i + \mathbf{K}(x)M_i(s_i)\rho_i(p_i)\mathbf{g}, \tag{1.6}$$

where $\alpha(s_1) = \frac{M_1(s_1)M_2(s_2)}{M(s_1)}\frac{dp_{12}}{ds}(s_1) \geqslant 0$, which completes the writing of the system in variables s_1 and p, except for the densities ρ_i.

The density of fluid depends on the pressure of the corresponding fluid. In this general case, the mathematical analysis is an open problem. A classical and physical assumption is to consider the densities to depend only on the global pressure. As a matter of fact, we take advantage of the fact that the capillary pressure is low and the densities vary slowly with the pressure (see [4, Chapter 4], for more details), we then assume that $\rho_i = \rho_i(p)$.

In this case, with (1.6), the system (1.1)–(1.4) can be rewritten as

$$\begin{aligned} &\phi\partial_t\big(\rho_i(p)s_i\big) - \operatorname{div}\big(\mathbf{K}\rho_i(p)M_i(s_i)\nabla p\big) - \operatorname{div}\big(\mathbf{K}\rho_i(p)\alpha(s_1)\nabla s_i\big) \\ &\quad + \operatorname{div}\big(\mathbf{K}M_i(s_i)\rho_i^2(p_i)\mathbf{g}\big) + \rho_i(p)s_i f_P = \rho_i(p)s_i^I f_I, \quad i = 1, 2, \end{aligned} \tag{1.7}$$

with (1.3).

This system is strongly nonlinear and highly coupled, namely by the evolution terms. It is possible to exhibit the evolution of pressure and of saturation. The saturation equation is given in (4.8), this equivalent system loses the conservative property and makes the analysis more complicated. In the particular case of pressure exponential law of densities, with the same compressibility factor for the two densities, the system can all the same be simplified and the analysis is proposed in [7] when some quadratic terms on velocities are neglected. Here, we study the case of bounded general densities, without simplification on nonlinear terms. The bound assumption on densities is validate by a maximum principle on pressure.

The system (1.7) contains further degeneracies. The capillary term α can vanish when $s_1 = 0$ and/or $s_1 = 1$ in physical situations. The lack of coercivity of the degenerate diffusion term $\operatorname{div}(\mathbf{K}\rho_i(p)\alpha(s_1)\nabla s_i)$ has been studied for incompressible flows [4–6], and for compressible flows [7], or mix compressible, incompressible flows [8,9]. One of the goals of this paper is to handle this degeneracy.

The dissipative terms on pressure $\operatorname{div}(\mathbf{K}\rho_i(p)M_i(s_i)\nabla p)$ degenerate when $s_i = 0$ due to the immobility of ith phase when it is missing. But, roughly speaking, where one mobility vanishes, the other is positive, finally, the combination of two degenerate dissipative estimates on pressure will lead to one non-degenerate estimate on pressure. The degeneracy of terms on pressure is overcome by a non-degenerate estimate on the gradient of pressure.

An additional difficulty is due to the degeneracy of the time derivative terms $\phi(x)\partial_t(\rho_i(p)s_i)$ whose the evolution term on pressure vanishes in the region where $s_i = 0$ for $i = 1, 2$. Nevertheless, the monotony of the densities ρ_1, ρ_2 ensures that the map $(\rho_1(p)s_1, \rho_2(p)s_2) \mapsto (s_1, p)$ is a diffeomorphism. This point is crucial to define the pressure along time and compactness property on the variable (s_1, p).

Finally, from the two degenerate evolution terms $\phi(x)\partial_t(\rho_i(p)s_i)$ $(i = 1, 2)$, follows a non-degenerate evolution of pressure. This is not the case for water gas flows studied in [8,9] since the constant density of one fluid lead to a degenerate evolution term on pressure on one hand, and on the other hand, allows a simpler equation on evolution of the saturation.

In this paper, the growth of densities with respect to pressure play a major role in the analysis and consequently the results does not include the case of mix compressible incompressible flows as water–gas problem, treated in [9].

We detail the physical context by introducing the boundary conditions, the initial conditions and some assumptions on the data of the problem.

Let $T > 0$, fixed and let Ω be a bounded set of $\mathbb{R}^d$ ($d \geqslant 1$). We set $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. To the system (1.7) ($i = 1,2$), we add the following mixed boundary conditions and initial conditions. We consider the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_{\mathrm{imp}}$, where Γ_1 denotes the injection boundary of the second phase and Γ_{imp} the impervious one,

$$\begin{cases} s_1(t,x) = 0, \quad p(t,x) = 0 & \text{on } \Gamma_1, \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 = 0 & \text{on } \Gamma_{\mathrm{imp}}, \end{cases} \tag{1.8}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_{imp}. We force a constant pressure (shifted at zero) along the time on the region of injection. This injection condition $s_1(t,x) = 0$ on Γ_1 is the first asymmetric property of the model. We could choose $s_2(t,x) = 0$, the result would be the same.

The initial conditions are defined on pressure and saturation

$$\begin{cases} p(0,x) = p^0(x) & \text{in } \Omega, \\ s_1(0,x) = s_1^0(x) & \text{in } \Omega. \end{cases} \tag{1.9}$$

Next we are going to introduce some physically relevant assumptions on the coefficients of the system.

(H1) The porosity $\phi \in W^{1,\infty}(\Omega)$ and there is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leqslant \phi(x) \leqslant \phi_1$ almost everywhere $x \in \Omega$.

(H2) The tensor $\mathbf{K}$ belongs to $(W^{1,\infty}(\Omega))^{d\times d}$. Moreover, there exist two positive constants k_0 and k_∞ such that

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leqslant k_\infty \quad \text{and} \quad \big(\mathbf{K}(x)\xi, \xi\big) \geqslant k_0|\xi|^2$$
$$\big(\text{for all } \xi \in \mathbb{R}^d,\ \text{almost everywhere } x \in \Omega\big).$$

(H3) The functions M_1 and M_2 belong to $\mathcal{C}^0([0,1];\mathbb{R}^+)$, $M_1(s_1 = 0) = 0$ and $M_2(s_2 = 0) = 0$. In addition, there is a positive constant m_0, such that, for all $s_1 \in [0,1]$,

$$M_1(s_1) + M_2(s_2) \geqslant m_0.$$

(H4) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geqslant 0$ almost everywhere $(t,x) \in Q_T$, $s_i^I(t,x) \geqslant 0$ ($i = 1,2$) and $s_1^I(t,x) + s_2^I(t,x) = 1$ almost everywhere in $(t,x) \in Q_T$.

(H5) The densities ρ_i ($i = 1,2$) are $\mathcal{C}^2(\mathbb{R})$ and increasing, there exist $\rho_m > 0$, $\rho_M > 0$ such that $\rho_m \leqslant \rho_i(p) \leqslant \rho_M$, for all p.

(H6) The function $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ satisfies $\alpha(s) > 0$ for $0 < s \leqslant 1$, and $\alpha(0) = 0$. There exist $r_1 > 0$, a_0 and $A_0 > 0$ such that, for all $s \in [0,1]$, $a_0 s^{r_1} \leqslant \alpha(s) \leqslant A_0 s^{r_1}$.

(H7) The function $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ satisfies $\alpha(s) > 0$ for $0 < s < 1$, and $\alpha(0) = \alpha(1) = 0$. There exist $r_1 > 0$, $0 < r_2 \leqslant 2$, $\xi^\star < 1$, a_0 and $A_0 > 0$ such that

$$a_0 s^{r_1} \leqslant \alpha(s) \leqslant A_0 s^{r_1}, \quad \text{for all } s \in [0,\xi^\star],$$
$$a_0(1-s)^{r_2} \leqslant \alpha(s) \leqslant A_0(1-s)^{r_2}, \quad \text{for all } s \in [\xi^\star,1].$$

Furthermore, there exist $q \geqslant r_2/2 + 1$ and a positive constant c such that the mobilities satisfy

$$M_1(s_1) \leqslant cs_1, \quad \forall s_1 \in [0, 1],$$
$$M_2(s_2) \leqslant cs_2^q, \quad \forall s_2 \in [0, 1].$$

The assumptions (H1)–(H4) are classical for porous media.

The assumption (H5) is a cut-off assumption on the density functions ρ_i which is valid when the pressure remains bounded. That is the case when $f_P = f_I$, according to Proposition 1.1.

The assumption (H6) or (H7) deals with the degeneracy of the function α. From a physical point of view, α can degenerate when a single phase is present (see [4, Chapter V]), that is "$s = 0$" or "$s = 1$." We investigate first the case where the degeneracy occurs only with one of the two phases. We choose to call "phase 1" such a phase, that is $\alpha(s_1 = 0) = 0$, $\alpha(s_1 = 1) > 0$. Such a degeneracy is detailed in assumption (H6) and can be referred as a "single" degeneracy. No limitation on the degeneracy is required for "single" degeneracy.

We also investigate the case of a "double" degeneracy, that is when $\alpha(s_1 = 0) = \alpha(s_1 = 1) = 0$ in assumption (H7). By convention, we choose to denote "phase 2," the one leading to the biggest degeneracy when such a phase is exclusively present ($s_2 = 1$). As a matter of fact, in (H7), the degeneracy of α around $s_1 = 0$ has no limitation whereas around $s_1 = 1$, the degeneracy is not allowed to be stronger than quadratic.

In assumption (H7), the degeneracy of the capillary term α is compared to polynomial growth and goes with degeneracy of the mobilities M_i.

The main two existence results are given below. The first one deals with a single and strong degeneracy. The second one treats the case of a double degeneracy, one strong and one weak (quadratic). For such a double degeneracy, we assume that no injection occurs ($f_I = 0$) and initially, the saturation is constrained.

Define

$$\beta(s) = \int_0^s \alpha(z)\, dz, \tag{1.10}$$

we can see that β^{-1} is a Hölder function of order θ, with $0 < \theta \leqslant 1$, on $[0, \beta(1)]$. Which means there exists a positive c such that for all $z_1, z_2 \in [0, \beta(1)]$, one has $|\beta^{-1}(z_1) - \beta^{-1}(z_2)| \leqslant c|z_1 - z_2|^\theta$. This is a consequence of assumption (H6) or (H7).

Let us define the following Sobolev space

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_1\},$$

this is an Hilbert space when equipped with the norm $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.

Let us state the main results of this paper.

Theorem 1.1 *(Strong single degeneracy). Let* (H1)–(H5) *and* (H6) *hold. Let initial data* (*introduced in* (1.9)) $0 \leqslant s_1^0 \leqslant 1$, p^0 *be defined almost everywhere in* Ω. *Then, there exists* (s_1, p) *satisfying*

$$0 \leqslant s_1 \leqslant 1 \quad \textit{a.e. in } Q_T, \qquad \beta(s_1) \in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{1.11}$$

$$p \in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \qquad \phi\partial_t\big(\rho_i(p)s_i\big) \in L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big) \quad (i=1,2), \tag{1.12}$$

such that for all $\varphi, \xi \in C^1([0,T];H^1_{\Gamma_1}(\Omega))$ *with* $\varphi(T) = \xi(T) = 0$,

$$\begin{aligned}
&-\int_{Q_T} \phi\rho_1(p)s_1\partial_t\varphi\,dx\,dt - \int_\Omega \phi(x)\rho_1\big(p^0(x)\big)s_1^0(x)\varphi(0,x)\,dx \\
&\quad + \int_{Q_T} \rho_1(p)M_1(s_1)\mathbf{K}\nabla p\cdot\nabla\varphi\,dx\,dt + \int_{Q_T} \mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_1\cdot\nabla\varphi\,dx\,dt \\
&\quad - \int_{Q_T} \mathbf{K}\rho_1^2(p)M_1(s_1)\mathbf{g}\cdot\nabla\varphi\,dx\,dt + \int_{Q_T} \rho_1(p)s_1 f_P\varphi\,dx\,dt \\
&= \int_{Q_T} \rho_1(p)s_1^I f_I\varphi\,dx\,dt,
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
&-\int_{Q_T} \phi\rho_2(p)s_2\partial_t\xi\,dx\,dt - \int_\Omega \phi(x)\rho_2\big(p^0(x)\big)s_2^0(x)\xi(0,x)\,dx \\
&\quad + \int_{Q_T} \rho_2(p)M_2(s_2)\mathbf{K}\nabla p\cdot\nabla\xi\,dx\,dt + \int_{Q_T} \mathbf{K}\rho_2(p)\alpha(s_1)\nabla s_2\cdot\nabla\xi\,dx\,dt \\
&\quad - \int_{Q_T} \mathbf{K}\rho_2^2(p)M_2(s_2)\mathbf{g}\cdot\nabla\xi\,dx\,dt + \int_{Q_T} \rho_2(p)s_2 f_P\xi\,dx\,dt \\
&= \int_{Q_T} \rho_2(p)s_2^I f_I\xi\,dx\,dt,
\end{aligned} \tag{1.14}$$

and finally the initial conditions are satisfied in a weak sense as follows:

$$\textit{for all } \psi \in H^1_{\Gamma_1}(\Omega) \textit{ the function } t \to \int_\Omega \phi\rho_i(p)s_i\psi\,dx \in \mathcal{C}^0\big([0,T]\big), \tag{1.15}$$

furthermore we have

$$\bigg(\int_\Omega \phi\rho_i(p)s_i\psi\,dx\bigg)(0) = \int_\Omega \phi\rho_i\big(p^0\big)s_i^0\psi\,dx. \tag{1.16}$$

Theorem 1.2 *(Strong and weak degeneracies). Let* $f_I = 0$. *Assume* (H1)–(H5) *and* (H7) *hold. Let initial data* (*introduced in* (1.9)) $0 \leqslant s_1^0 \leqslant 0$, p^0 *be defined almost everywhere in* Ω *and the initial saturation verifies*

$$G\big(s_1^0\big) \in L^1(\Omega),$$

where the function G is defined by (4.6).

Then, there exists (s_1, p) *satisfying*

$$0 \leqslant s_1 \leqslant 1 \quad \textit{a.e. in } Q_T, \qquad G(s_1) \in L^\infty\big(0,T; L^1(\Omega)\big),$$
$$\beta(s_1) \in L^2\big(0,T; H^1_{\Gamma_1}(\Omega)\big), \tag{1.17}$$
$$p \in L^2\big(0,T; H^1_{\Gamma_1}(\Omega)\big), \qquad \phi\partial_t\big(\rho_i(p)s_i\big) \in L^2\big(0,T; \big(H^1_{\Gamma_1}(\Omega)\big)'\big) \quad (i=1,2), \tag{1.18}$$

such that the formulations (1.13)*,* (1.14) *are satisfied and initial conditions verify* (1.15) *and* (1.16).

As we can see, the above notion of weak solutions is very natural provided that we explain the origin of the requirements (1.11)–(1.12) or (1.17)–(1.18). Obviously, they correspond to *a priori* estimates. Indeed, (1.13)–(1.14) ensure that $s_i \geqslant 0$ $(i = 1, 2)$ which is equivalent to $0 \leqslant s_i \leqslant 1$ (the proof is detailed in Proposition 2.2). The key point is to obtain the estimate on ∇p.

For that, define $g_1(p) = \int_0^p \rho_2(q)\,dq$ and $g_2(p) = \int_0^p \rho_1(q)\,dq$ and note that at least formally:

$$\partial_t\big(\rho_1(p)s_1\big)g_1(p) + \partial_t\big(\rho_2(p)s_2\big)g_2(p)$$
$$= \rho_1'(p)g_1(p)s_1\partial_t p + \rho_1(p)g_1(p)\partial_t s_1 + \rho_2'(p)g_2(p)s_2\partial_t p + \rho_2(p)g_2(p)\partial_t s_2.$$

The functions $\mathcal{H}_1$ and $\mathcal{H}_2$ defined by

$$\mathcal{H}_1(p) = \rho_1(p)g_1(p) - \int_0^p \rho_1(q)\rho_2(q)\,dq, \tag{1.19}$$

$$\mathcal{H}_2(p) = \rho_2(p)g_2(p) - \int_0^p \rho_1(q)\rho_2(q)\,dq, \tag{1.20}$$

satisfy $\mathcal{H}_i'(p) = \rho_i'(p)g_i(p)$, $\mathcal{H}_i(0) = 0$, $\mathcal{H}_i(p) \geqslant 0$ for all p, and $\mathcal{H}_i$ is sublinear with respect to p. The identity $s_1 + s_2 = 1$ implies

$$\partial_t\big(\rho_1(p)s_1\big)g_1(p) + \partial_t\big(\rho_2(p)s_2\big)g_2(p) = \partial_t\big(s_1\mathcal{H}_1(p)\big) + \partial_t\big(s_2\mathcal{H}_2(p)\big).$$

Taking $\varphi = g_1(p)$ in (1.13) and $\xi = g_2(p)$ in (1.14), the sum leads to

$$\frac{d}{dt}\int_\Omega \phi s_1\mathcal{H}_1(p)\,dx + \frac{d}{dt}\int_\Omega \phi s_2\mathcal{H}_2(p)\,dx$$
$$+ \int_\Omega \rho_1(p)\rho_2(p)\big(M_1(s_1) + M_2(s_2)\big)\mathbf{K}\nabla p\cdot\nabla p\,dx$$
$$- \int_\Omega \mathbf{K}\rho_1(p)\rho_2(p)\big(\rho_1(p)M_1(s_1) + \rho_2(p)M_2(s_2)\big)\mathbf{g}\cdot\nabla p\,dx$$

$$+\int_\Omega \big(\rho_1(p)g_1(p)s_1+\rho_2(p)g_2(p)s_2\big) f_P\,dx$$

$$=\int_\Omega \big(\rho_1(p)g_1(p)s_1^I+\rho_2(p)g_2(p)s_2^I\big) f_I\,dx. \tag{1.21}$$

Using the assumptions (H3) and (H5), and the fact that the functions $\mathcal{H}_i$ are non-negative and sublinear with respect to p, we deduce $p\in L^2(0,T;H^1_{\Gamma_1}(\Omega))$.

Due to the degeneracy of the capillary function α, the bound on $\beta(s_1)$ in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$ is too technical and we refer to Lemmas 4.1 and 4.2 for the estimates. When α is not degenerated, s_1 belongs to $L^2(0,T;H^1_{\Gamma_1}(\Omega))$, this estimate is trivial and is obtained by taking $\varphi=\rho_1(p)s_1$ in (1.13).

The next result makes the cut-off assumption (H5) physically relevant.

Proposition 1.1 *(Maximum principle on pressure). Let $f_I=f_P$ and $\mathbf{g}=\mathbf{0}$. If $p_{\min}\leqslant p_0\leqslant p_{\max}$ then $p_{\min}\leqslant p(t,x)\leqslant p_{\max}$, almost everywhere in (t,x) in Q_T.*

The proof of this proposition is based on energy estimates with regularized Heaviside functions of the shifted pressure.

Before establishing Theorems 1.1 and 1.2, we introduce the existence of solutions to system (1.7) under the conditions (H1)–(H5), when α is replaced by a non-degenerate positive function,

$$\alpha_\eta(s)=\alpha(s)+\eta,\qquad \eta>0,$$

then we consider the non-degenerate system

$$\begin{aligned}&\phi\partial_t\big(\rho_i\big(p^\eta\big)s_i^\eta\big)-\operatorname{div}\big(\mathbf{K}\rho_i\big(p^\eta\big)M_i\big(s_i^\eta\big)\nabla p^\eta\big)-\operatorname{div}\big(\mathbf{K}\rho_i\big(p^\eta\big)\alpha_\eta\big(s_1^\eta\big)\nabla s_i^\eta\big)\\&\quad+\operatorname{div}\big(\mathbf{K}M_i\big(s_i^\eta\big)\rho_i^2\big(p^\eta\big)\mathbf{g}\big)+\rho_i\big(p^\eta\big)s_i^\eta f_P=\rho_i\big(p^\eta\big)s_i^I f_I,\end{aligned} \tag{1.22}$$

completed with boundary conditions (1.8) and initial conditions (1.9).

Existence of solutions of system (1.22) is given in the following theorem.

Theorem 1.3 *(Non-degenerate dissipative model). Let* (H1)–(H5) *hold. Let $s_i^0\geqslant 0$, p^0 (introduced in* (1.9)*) be defined almost everywhere in Ω. Then, for all $\eta>0$, there exists (s_1^η,p^η) satisfying*

$$\begin{gathered}0\leqslant s_i^\eta(t,x)\leqslant 1\quad \textit{a.e. in } Q_T,\qquad s_i^\eta\in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big),\quad i=1,2,\\ p^\eta\in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big),\qquad \phi\partial_t\big(\rho_i\big(p^\eta\big)s_i^\eta\big)\in L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big),\\ \rho_i\big(p^\eta\big)s_i^\eta\in C^0\big(0,T;L^2(\Omega)\big),\quad i=1,2,\end{gathered}$$

for all $\varphi\in L^2(0,T;H^1_{\Gamma_1}(\Omega))$,

$$\langle \phi\partial_t(\rho_i(p^\eta)s_i^\eta), \varphi\rangle + \int_{Q_T} \rho_i(p^\eta)M_i(s_i^\eta)\mathbf{K}\nabla p^\eta \cdot \nabla\varphi\,dx\,dt$$
$$+ \int_{Q_T} \mathbf{K}\rho_i(p^\eta)\alpha_\eta(s_1^\eta)\nabla s_i^\eta \cdot \nabla\varphi\,dx\,dt - \int_{Q_T} \mathbf{K}\rho_i^2(p^\eta)M_i(s_i^\eta)\mathbf{g} \cdot \nabla\varphi\,dx\,dt$$
$$+ \int_{Q_T} \rho_i(p^\eta)s_i^\eta f_P\varphi\,dx\,dt = \int_{Q_T} \rho_i(p^\eta)s_i^I f_I\varphi\,dx\,dt, \quad i = 1, 2. \tag{1.23}$$

The bracket $\langle\cdot,\cdot\rangle$ *denotes the duality product between* $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ *and* $L^2(0,T; H^1_{\Gamma_1}(\Omega))$.

The sequel of the article is organized as follows. In the next section, we construct approached solutions solving a time discretization problem of (1.23). This choice is motivated by the fact that no evolution have to be considered in a first step. The problem of degeneracy of evolution term is temporarily seated aside. Furthermore, the maximum principle is conserved on saturation. This requires the study of an auxiliary nonlinear elliptic problem in order to well define time discrete solutions. The next section is devoted to the analysis of this elliptic problem. For the same reasons, this technique had been employed in [9] for water gas flows.

Section 3 is devoted to the passage of a discrete problem to a continuous one and establishes Theorem 1.3. Then, in Section 4, it remains to pass to the limit on the parameter η to treat the case of a degenerate capillary term. Finally, we let the proof of the maximum principle on pressure in the last section.

2. Study of a nonlinear elliptic system

Having in mind a time discretization of (1.23), we are concerned with the following system,

$$\phi\frac{\rho_i(p)s_i - \rho_i^\star s_i^\star}{h} - \operatorname{div}\big(\mathbf{K}\rho_i(p)M_i(s_i)\nabla p\big) - \operatorname{div}\big(\mathbf{K}\rho_i(p)\alpha_\eta(s_1)\nabla s_i\big)$$
$$+ \operatorname{div}\big(\mathbf{K}M_i(s_i^\eta)\rho_i^2\mathbf{g}\big) + \rho_i(p)s_i f_P = \rho_i(p)s_i^I f_I. \tag{2.1}$$

We introduce two regularizations, we add a second order operator in pressure in order to reinforce the ellipticity of the system, we also trunk high frequencies of nonlinear elliptic term in pressure in Eq. (2.1). Let $\mathcal{P}_N$ be the orthogonal projector of $H^1_{\Gamma_1}(\Omega)$ on the first N eigenvectors of the operator

$$p \to -\Delta p$$

with homogeneous Dirichlet boundary conditions. Such perturbations lead to the loss of maximum principle on the saturation s, so the functions M_1, M_2 and α are extended on $\mathbb{R}$ by continuous constant functions outside [0, 1] and then are bounded on $\mathbb{R}$. For the same reason we denote

$$Z(s) = \begin{cases} 0 & \text{for } s \leqslant 0, \\ s & \text{for } s \in [0; 1], \\ 1 & \text{for } s \geqslant 1. \end{cases}$$

Existence of solutions to (2.1) is constructed in three steps. The first one consists in studying the following problem for fixed parameters $N > 0$ and $\varepsilon > 0$,

$$\int_\Omega \phi \frac{\rho_1(p^{N,\varepsilon})Z(s_1^{N,\varepsilon}) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \mathbf{K}\rho_1\big(p^{N,\varepsilon}\big)M_1\big(s_1^{N,\varepsilon}\big)\rho_2\big(p^{N,\varepsilon}\big)\nabla\mathcal{P}_N p^{N,\varepsilon}\cdot\nabla\varphi\,dx$$
$$+\int_\Omega \mathbf{K}\rho_1\big(p^{N,\varepsilon}\big)\alpha_\eta\big(s_1^{N,\varepsilon}\big)\nabla s_1^{N,\varepsilon}\cdot\nabla\varphi\,dx - \int_\Omega \mathbf{K}\rho_1^2\big(p^{N,\varepsilon}\big)M_1\big(s_1^{N,\varepsilon}\big)\mathbf{g}\cdot\nabla\varphi\,dx$$
$$+\int_\Omega \rho_1\big(p^{N,\varepsilon}\big)Z\big(s_1^{N,\varepsilon}\big)f_P\varphi\,dx = \int_\Omega \rho_1\big(p^{N,\varepsilon}\big)s_1^I f_I\varphi, \tag{2.2}$$

$$\int_\Omega \phi \frac{\rho_2(p^{N,\varepsilon})Z(s_2^{N,\varepsilon}) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \mathbf{K}\rho_2\big(p^{N,\varepsilon}\big)\big(M_2\big(s_2^{N,\varepsilon}\big)+\varepsilon\big)\nabla p^{N,\varepsilon}\cdot\nabla\xi\,dx$$
$$+\int_\Omega \mathbf{K}\rho_2\big(p^{N,\varepsilon}\big)\alpha_\eta\big(s_1^{N,\varepsilon}\big)\nabla s_2^{N,\varepsilon}\cdot\nabla\xi\,dx - \int_\Omega \mathbf{K}\rho_2^2\big(p^{N,\varepsilon}\big)M_2\big(s_2^{N,\varepsilon}\big)\mathbf{g}\cdot\nabla\xi\,dx$$
$$+\int_\Omega \rho_2\big(p^{N,\varepsilon}\big)Z\big(s_2^{N,\varepsilon}\big)f_P\xi\,dx = \int_\Omega \rho_2\big(p^{N,\varepsilon}\big)s_2^I f_I\xi\,dx, \tag{2.3}$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.

Remark that the two additional regularizations are parametrized by N and ε. The projector $\mathcal{P}_N$ appears in (2.2) to make regular the implied term. It appears twice in order to conserve a coercivity property, as for the non-regularized term. The second regularization reinforce the diffusive term on pressure in (2.3) since a part of the diffusion on pressure is lost in (2.2) because of the operator $\mathcal{P}_N$. The symmetries in (2.2)–(2.3) are voluntary lost in order to consider an equation on saturation (2.2), thus an equation on pressure (2.3) for some fixed variables in these systems.

The second step concerns the passage to the limit as N goes to infinity in order to recover the full physical diffusion on pressure. Finally, the last one is the passage to the limit as ε goes to zero.

Step 1. We show for fixed $N > 0$ and $\varepsilon > 0$ existence of solutions to (2.2)–(2.3). We omit for the time being the dependence of solutions on parameters N and ε.

Proposition 2.1. *Assume $\rho_i^\star s_i^\star$ belongs to $L^2(\Omega)$. Then, there exists (s_1, p) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$, solution of* (2.2)–(2.3).

Proof. The proof is based on the Leray–Schauder fixed point theorem.

Let $\mathcal{T}$ be a map from $L^2(\Omega)$ to $L^2(\Omega)$ defined by

$$\mathcal{T}(\bar{s}_1,\bar{p}) = (s_1, p),$$

where the pair (s_1, p) is the unique solution of the system (2.4)–(2.5)

$$\int_\Omega \phi \frac{\rho_1(\overline{p})Z(\bar{s}_1) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \rho_1(\overline{p})M_1(\bar{s}_1)\mathbf{K}\nabla\mathcal{P}_N\overline{p}\cdot\nabla\varphi\,dx$$
$$+ \int_\Omega \mathbf{K}\rho_1(\overline{p})\alpha_\eta(\bar{s}_1)\nabla s_1\cdot\nabla\varphi\,dx - \int_\Omega \rho_1^2(\overline{p})M_1(\bar{s}_1)\mathbf{Kg}\cdot\nabla\varphi\,dx$$
$$+ \int_\Omega \rho_1(\overline{p})Z(\bar{s}_1)f_P\varphi\,dx = \int_{Q_T} \rho_1(\overline{p})s_1^I f_I\varphi\,dx, \tag{2.4}$$

$$\int_\Omega \phi \frac{\rho_2(\overline{p})Z(\bar{s}_2) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \rho_2(\overline{p})\big(M_2(\bar{s}_2)+\varepsilon\big)\mathbf{K}\nabla p\cdot\nabla\xi\,dx$$
$$+ \int_\Omega \mathbf{K}\rho_2(\overline{p})\alpha_\eta(\bar{s}_1)\nabla s_2\cdot\nabla\xi\,dx + \int_\Omega \rho_2(\overline{p})Z(\bar{s}_2)f_P\xi\,dx = \int_{Q_T}\rho_2(\overline{p})s_2^I f_I\xi\,dx \tag{2.5}$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.

As a matter of fact, the map $\mathcal{T}$ is well defined on $L^2(\Omega)$ by using Lax–Milgram theorem in (2.5) to define a unique $s_1\in H^1_{\Gamma_1}(\Omega)$, then, using again Lax–Milgram theorem in (2.4), we have a unique solution $p\in H^1_{\Gamma_1}(\Omega)$.

Lemma 2.1. *The map $\mathcal{T}$ is a continuous operator which maps every bounded subset of $L^2(\Omega)$ into a relatively compact set.*

Proof. Let us consider a sequence $(\bar{s}_{1,n},\overline{p}_n)$ of a bounded set of $L^2(\Omega)\times L^2(\Omega)$ which converges to $(\bar{s}_1,\overline{p})\in L^2(\Omega)\times L^2(\Omega)$. Let us prove that $(s_{1,n},p_n)=\mathcal{T}(\bar{s}_{1,n},\overline{p}_n)$ is bounded in $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$ and converges to $(s_1,p)=\mathcal{T}(\bar{s}_1,\overline{p})$.

The sequence $(s_{1,n},p_n)$ verifies

$$\int_\Omega \phi \frac{\rho_1(\overline{p}_n)Z(\bar{s}_{1,n}) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \rho_1(\overline{p}_n)M_1(\bar{s}_{1,n})\mathbf{K}\nabla\mathcal{P}_N\overline{p}_n\cdot\nabla\varphi\,dx$$
$$+ \int_\Omega \mathbf{K}\rho_1(\overline{p}_n)\alpha_\eta(\bar{s}_{1,n})\nabla s_{1,n}\cdot\nabla\varphi\,dx - \int_\Omega \rho_1^2(\overline{p}_n)M_1(\bar{s}_{1,n})\mathbf{Kg}\cdot\nabla\varphi\,dx$$
$$+ \int_\Omega \rho_1(\overline{p}_n)Z(\bar{s}_{1,n})f_P\varphi\,dx = \int_\Omega \rho_1(\overline{p}_n)s_1^I f_I\varphi\,dx, \tag{2.6}$$

$$\int_\Omega \phi \frac{\rho_2(\overline{p}_n)Z(\bar{s}_{2,n}) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \rho_2(\overline{p}_n)\big(M_2(\bar{s}_{2,n}) + \varepsilon\big)\mathbf{K}\nabla p_n \cdot \nabla\xi\,dx$$
$$+ \int_\Omega \mathbf{K}\rho_2(\overline{p}_n)\alpha_\eta(\bar{s}_{1,n})\nabla s_{2,n} \cdot \nabla\xi\,dx - \int_\Omega \rho_2^2(\overline{p}_n)M_2(\bar{s}_{2,n})\mathbf{K}\mathbf{g} \cdot \nabla\xi\,dx$$
$$+ \int_\Omega \rho_2(\overline{p}_n)Z(\bar{s}_{2,n})f_P\xi\,dx = \int_\Omega \rho_2(\overline{p}_n)s_2^I f_I\xi\,dx \tag{2.7}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Let us take $\varphi = s_{1,n} \in H^1_{\Gamma_1}(\Omega)$ in (2.6), we get

$$k_0\eta\rho_m \int_\Omega |\nabla s_{1,n}|^2\,dx \leqslant C + C\|s_{1,n}\|^2_{L^2(\Omega)} + C\|\nabla\mathcal{P}_N\overline{p}_n\|^2_{L^2(\Omega)}, \tag{2.8}$$

where C depends on h, ϕ_1, $\|f_P\|_{L^2(\Omega)}$, ρ_M, k_∞ and $\|\rho^\star s^\star\|_{L^2(\Omega)}$. As

$$\|\nabla\mathcal{P}_N\overline{p}_n\|_{L^2(\Omega)} \leqslant c_N\|\overline{p}_n\|_{L^2(\Omega)},$$

the Poincaré inequality and the estimate (2.8) ensure that the sequence $(s_{1,n})_n$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$.

Then, taking $\xi = p_n$ in (2.7), we get

$$\varepsilon k_0 \int_\Omega |\nabla p_n|^2\,dx \leqslant C\big(1 + \|\nabla s_{1,n}\|^2_{L^2(\Omega)}\big) + C\|p_n\|_{L^2(\Omega)},$$

where C depends on h, ϕ_1, $\|f_P\|_{L^2(\Omega)}$, $\|f_I\|_{L^2(\Omega)}$, k_∞ and $\|\alpha_\eta\|_{L^\infty(\mathbb{R})}$. Then the sequence $(p_n)_n$ is also uniformly bounded in $H^1_{\Gamma_1}(\Omega)$. This establishes the relative compactness property of the map $\mathcal{T}$.

Furthermore, up to a subsequence, we have the convergences

$$s_{1,n} \to s_1 \quad \text{weakly in } H^1_{\Gamma_1}(\Omega), \tag{2.9}$$
$$p_n \to p \quad \text{weakly in } H^1_{\Gamma_1}(\Omega), \tag{2.10}$$
$$s_{1,n} \to s_1 \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{2.11}$$
$$p_n \to p \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{2.12}$$

Passage to the limit in (2.6). Almost everywhere convergence of $\rho_1(\overline{p}_n)\varphi$ dominated by $\rho_M|\varphi|$ and (2.11) allow to apply the Lebesgue theorem and pass to the limit of the first term of (2.6). The second term is treated as follows. The sequence $(M_1(\bar{s}_{1,n})\rho_1(\overline{p}_n)\nabla\mathcal{P}_N\varphi)_n$ is dominated and converges a.e. as n goes to infinity. Then, by Lebesgue theorem, we have

$$M_1(\bar{s}_{1,n})\rho_1(\overline{p}_n)\nabla\varphi \to M_1(\bar{s})\rho_1(\overline{p})\nabla\varphi. \tag{2.13}$$

Furthermore $\overline{p}_n$ converges in $L^2(\Omega)$, it follows that

$$\nabla \mathcal{P}_N \overline{p}_n \to \nabla \mathcal{P}_N \overline{p} \quad \text{strongly in } \left(L^2(\Omega)\right)^d. \tag{2.14}$$

Then, the convergences (2.13)–(2.14) establish the limit for the second term.

In the same way,

$$\rho_1(\overline{p}_n)\alpha_\eta(\bar{s}_{1,n})\nabla\varphi \to \rho_1(\overline{p})\alpha_\eta(\bar{s}_1)\nabla\varphi \quad \text{strongly in } \left(L^2(\Omega)\right)^d,$$

the convergence (2.9) allows to pass to the limit for the third term. The convergences of the last terms are always an application of the Lebesgue theorem.

The passage to the limit on (2.5) is obtained in the same manner.

Then (s_1, p) is solution to (2.4)–(2.5), which establishes the continuity and achieves the proof of the lemma. $\square$

Lemma 2.2 *(A priori estimate). There exists $r > 0$ such that, if $(s_1, p) = \lambda\mathcal{T}(s_1, p)$ with $\lambda \in (0, 1)$, then*

$$\left\|(s_1, p)\right\|_{L^2(\Omega)\times L^2(\Omega)} \leqslant r.$$

Proof. Assume $(s_1, p) = \lambda\mathcal{T}(s_1, p)$ exists, then (s_1, p) satisfies

$$\begin{aligned}
&\lambda\int_\Omega \phi\frac{\rho_1(p)Z(s_1) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \lambda\int_\Omega \rho_1(p)M_1(s_1)\mathbf{K}\nabla\mathcal{P}_N p\cdot\nabla\varphi\,dx\\
&\quad+\int_\Omega \mathbf{K}\rho_1(p)\alpha_\eta(s_1)\nabla s_1\cdot\nabla\varphi\,dx - \lambda\int_\Omega \rho_1^2(p)M_1(s_1)\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx\\
&\quad+\lambda\int_\Omega \rho_1(p)Z(s_1)f_P\varphi\,dx = \lambda\int_\Omega \rho_1(p)s_1^I f_I\varphi\,dx,
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
&\lambda\int_\Omega \phi\frac{\rho_2(p)Z(s_2) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \rho_2(p)\big(M_2(s_2)+\varepsilon\big)\mathbf{K}\nabla p\cdot\nabla\xi\,dx\\
&\quad+\int_\Omega \mathbf{K}\rho_2(p)\alpha_\eta(s_1)\nabla s_2\cdot\nabla\xi\,dx - \lambda\int_\Omega \rho_2^2(p)M_2(s_2)\mathbf{K}\mathbf{g}\cdot\nabla\xi\,dx\\
&\quad+\lambda\int_{Q_T} \rho_2(p)Z(s_2)f_P\xi\,dx = \lambda\int_\Omega \rho_2(p)s_2^I f_I\xi\,dx
\end{aligned}\tag{2.16}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Consider $\varphi = g_1(p) := \int_0^p \rho_2(q)\,dq \in H^1_{\Gamma_1}(\Omega)$ in (2.15) and $\xi = g_2(p) := \int_0^p \rho_1(q)\,dq \in H^1_{\Gamma_1}(\Omega)$ in (2.15). Summing these quantities, we obtain

$$\begin{aligned}
&\frac{\lambda}{h}\int_\Omega \phi\big(\big(\rho_1(p)Z(s_1)-\rho_1^\star s_1^\star\big)g_1(p)+\big(\rho_2(p)Z(s_2)-\rho_2^\star s_2^\star\big)g_2(p)\big)\,dx\\
&\qquad+\lambda\int_\Omega \rho_1(p)\rho_2(p)M_1(s_1)\mathbf{K}\nabla\mathcal{P}_N p\cdot\nabla\mathcal{P}_N p\,dx\\
&\qquad+\int_\Omega \rho_1(p)\rho_2(p)\big(M_2(s_2)+\varepsilon\big)\mathbf{K}\nabla p\cdot\nabla p\,dx\\
&\qquad-\lambda\int_\Omega \rho_1(p)\rho_2(p)\big(\rho_1(p)M_1(s_1)+\rho_2(p)M_2(s_2)\big)\mathbf{K}\mathbf{g}\cdot\nabla p\,dx\\
&\qquad+\lambda\int_\Omega \big(\rho_1(p)Z(s_1)g_1(p)+\rho_2(p)Z(s_2)g_2(p)\big)f_P\,dx\\
&\quad=\lambda\int_\Omega \big(\rho_1(p)s_1^I g_1(p)+\rho_2(p)s_2^I g_2(p)\big)f_I\,dx.
\end{aligned}\tag{2.17}$$

Remark that the functions $p \to g_i(p)$ is sublinear, the second integral on the left-hand side is non-negative, we deduce from Cauchy–Schwartz and Poincaré inequalities that (2.17) reduces to

$$\varepsilon\rho_m^2 k_0\int_\Omega |\nabla p|^2\,dx \leqslant C_1\big(1+\|f_P\|^2_{L^2(\Omega)}+\|f_I\|^2_{L^2(\Omega)}+\big\|\rho_1^\star s_1^\star\big\|^2_{L^2(\Omega)}+\big\|\rho_2^\star s_2^\star\big\|^2_{L^2(\Omega)}\big),\tag{2.18}$$

where C_1 depends on h and ε, but not on λ.

On the other hand, consider $\xi = -s_1 \in H^1_{\Gamma_1}(\Omega)$ in (2.16), we have

$$\begin{aligned}
&\int_\Omega \mathbf{K}\rho_2(p)\alpha_\eta(s_1)\nabla s_1\cdot\nabla s_1\,dx\\
&\quad=\frac{\lambda}{h}\int_\Omega \phi\big(\rho_2(p)Z(s_2)-\rho_2^\star s_2^\star\big)s_1\,dx+\lambda\int_\Omega \rho_2^2(p)M_2(s_2)\mathbf{K}\mathbf{g}\cdot\nabla s_1\,dx\\
&\qquad+\int_\Omega \big(M_2(s_2)+\varepsilon\big)\mathbf{K}\nabla p\cdot\nabla s_1\,dx+\lambda\int_\Omega \rho_2(p)Z(s_2)f_P s_1\,dx-\lambda\int_\Omega \rho_2(p)s_2^I f_I s_1\,dx.
\end{aligned}\tag{2.19}$$

The Cauchy–Schwartz inequality and the Poincaré inequality permit to obtain

$$\rho_m k_0 \eta \int_\Omega |\nabla s_1|^2\,dx \leqslant C_1 + C_2 \|\nabla p\|^2_{L^2(\Omega)} + C_3\big(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)}\big), \tag{2.20}$$

where constants C_i $(i = 1, 3)$ do not depend on λ, more precisely, C_1 depends on h and η, C_2 depends on ε and η, and C_3 depends on η. And from (2.18), we have

$$\|\nabla s_1\|_{L^2(\Omega)} \leqslant C_4, \tag{2.21}$$

and constant C_4 does not depend on λ.

Lemma 2.2 is then established. $\square$

Lemmas 2.1 and 2.2 allow to apply the Leray–Schauder fixed point theorem [13] and prove the existence of a solution to (2.2)–(2.3). This completes Proposition 2.1. $\square$

Step 2. Now we are concerned with the limit N goes to infinity (we omit the dependence of solutions on ε). For all N, we have established a solution $(p_N, s_N) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ to (2.2)–(2.3) satisfying

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho(p_N)Z(s_{1,N}) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \rho_1(p_N)M_1(s_{1,N})\mathbf{K}\nabla \mathcal{P}_N p_N \cdot \nabla\varphi\,dx \\
&\quad + \int_\Omega \mathbf{K}\rho_1(p_N)\alpha_\eta(s_{1,N})\nabla s_{1,N}\cdot\nabla\varphi\,dx - \int_\Omega \rho_1^2(p_N)M_1(s_{1,N})\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx \\
&\quad + \int_\Omega \rho_1(p_N)Z(s_{1,N})f_P\varphi\,dx = \int_\Omega \rho_1(p_N)s_1^I f_I\varphi\,dx,
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho(p_N)Z(s_{2,N}) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \rho_2(p_N)\big(M_2(s_{2,N}) + \varepsilon\big)\mathbf{K}\nabla p \cdot \nabla\xi\,dx \\
&\quad + \int_\Omega \mathbf{K}\rho_2(p_N)\alpha_\eta(s_{1,N})\nabla s_{2,N}\cdot\nabla\xi\,dx - \int_\Omega \rho_2^2(p_N)M_2(s_{2,N})\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx \\
&\quad + \int_\Omega \rho_2(p_N)Z(s_{2,N})f_P\xi\,dx = \int_\Omega \rho_2(p_N)s_2^I f_I\xi\,dx
\end{aligned} \tag{2.23}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Reproducing the estimates (2.18)–(2.21) with $\lambda = 1$, we deduce the same estimates which are uniform with N

$$\begin{gathered}
\varepsilon\rho_m^2 k_0 \|\nabla p_N\|^2_{L^2(\Omega)} \leqslant C_1\big(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \big\|\rho_1^\star s_1^\star\big\|^2_{L^2(\Omega)} + \big\|\rho_2^\star s_2^\star\big\|^2_{L^2(\Omega)}\big), \\
\|\nabla s_{1,N}\|_{L^2(\Omega)} \leqslant C_4.
\end{gathered}$$

Then, up to a subsequence, we have the convergences

$$s_{1,N} \to s_1 \qquad \text{weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{2.24}$$

$$p_N \to p \qquad \text{weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{2.25}$$

The convergences in (2.22)–(2.23) with respect to N are obtained in the same manner as for the convergences with respect to n in (2.6)–(2.7).

Step 3. Passage to the limit as ε goes to zero. For all $\varepsilon > 0$, we have shown that there exists $(s_{1,\varepsilon}, p_\varepsilon) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, satisfying

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho(p_\varepsilon)Z(s_{1,\varepsilon}) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \rho_1(p_\varepsilon)M_1(s_{1,\varepsilon})\mathbf{K}\nabla p_\varepsilon \cdot \nabla\varphi\,dx \\
&\quad + \int_\Omega \mathbf{K}\rho_1(p_\varepsilon)\alpha_\eta(s_{1,\varepsilon})\nabla s_{1,\varepsilon} \cdot \nabla\varphi\,dx - \int_\Omega \rho_1^2(p_\varepsilon)M_1(s_{1,\varepsilon})\mathbf{K}\mathbf{g} \cdot \nabla\varphi\,dx \\
&\quad + \int_\Omega \rho_1(p_\varepsilon)Z(s_{1,\varepsilon})f_P\varphi\,dx = \int_\Omega \rho_1(p_\varepsilon)s_1^I f_I\varphi\,dx,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho(p_\varepsilon)Z(s_{2,\varepsilon}) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \rho_2(p_\varepsilon)\big(M_2(s_{2,\varepsilon}) + \varepsilon\big)\mathbf{K}\nabla p_\varepsilon \cdot \nabla\xi\,dx \\
&\quad + \int_\Omega \mathbf{K}\rho_2(p_\varepsilon)\alpha_\eta(s_{1,\varepsilon})\nabla s_{2,\varepsilon} \cdot \nabla\xi\,dx - \int_\Omega \rho_2^2(p_\varepsilon)M_2(s_{2,\varepsilon})\mathbf{K}\mathbf{g} \cdot \nabla\xi\,dx \\
&\quad + \int_\Omega \rho_2(p_\varepsilon)Z(s_{2,\varepsilon})f_P\xi\,dx = \int_\Omega \rho_2(p_\varepsilon)s_2^I f_I\xi\,dx
\end{aligned} \tag{2.27}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Lemma 2.3 *(Uniform estimates with respect to ε). The solutions $(s_{1,\varepsilon}, p_\varepsilon)$ of* (2.26)–(2.27) *satisfy*

$$\int_\Omega |\nabla p_\varepsilon|^2\,dx\,dt \leqslant C_1, \tag{2.28}$$

$$\int_\Omega \alpha_\eta(s_{1,\varepsilon})|\nabla s_{1,\varepsilon}|^2\,dx\,dt \leqslant C_2, \tag{2.29}$$

where C_1 and C_2 depend only on Ω, ϕ_1, k_0, k_∞, $\|f_P\|_{L^2(\Omega)}$, $\|f_I\|_{L^2(\Omega)}$, $\|\rho_1^\star s_1^\star\|_{L^2(\Omega)}$, $\|\rho_2^\star s_2^\star\|_{L^2(\Omega)}$, ρ_m, ρ_M and h.

Proof. Consider again $\varphi = g_1(p) := \int_0^p \rho_2(q)\,dq \in H^1_{\Gamma_1}(\Omega)$ in (2.26) and $\xi = g_2(p) := \int_0^p \rho_1(q)\,dq \in H^1_{\Gamma_1}(\Omega)$ in (2.27). We sum these quantities to obtain

$$\frac{1}{h}\int_{\Omega}\phi\big(\big(\rho_1(p_\varepsilon)Z(s_{1,\varepsilon})-\rho_1^\star s_1^\star\big)g_1(p_\varepsilon)+\big(\rho_2(p_\varepsilon)Z(s_{2,\varepsilon})-\rho_2^\star s_2^\star\big)g_2(p_\varepsilon)\big)\,dx$$
$$+\int_{\Omega}\rho_1(p_\varepsilon)\rho_2(p_\varepsilon)\big(M_1(s_\varepsilon)+M_2(s_\varepsilon)+\varepsilon\big)\mathbf{K}\nabla p_\varepsilon\cdot\nabla p_\varepsilon\,dx$$
$$-\int_{\Omega}\rho_1(p_\varepsilon)\rho_2(p_\varepsilon)\big(\rho_1(p_\varepsilon)M_1(s_\varepsilon)+\rho_2(p_\varepsilon)M_2(s_\varepsilon)\big)\mathbf{K}\mathbf{g}\cdot\nabla p_\varepsilon\,dx$$
$$+\int_{\Omega}\big(\rho_1(p_\varepsilon)Z(s_{1,\varepsilon})g_1(p_\varepsilon)+\rho_2(p_\varepsilon)Z(s_{2,\varepsilon})g_2(p_\varepsilon)\big)f_P\,dx$$
$$=\int_{\Omega}\big(\rho_1(p_\varepsilon)s_1^I g_1(p_\varepsilon)+\rho_2(p_\varepsilon)s_2^I g_2(p_\varepsilon)\big)f_I\,dx. \tag{2.30}$$

This estimate involves a uniform dissipative term on pressure since

$$M_1(s_\varepsilon)+M_2(s_\varepsilon)+\varepsilon\geqslant m_0,$$

according to assumption (H3).

We recall that $p_\varepsilon\mapsto g_i(p_\varepsilon)$ is sublinear (i.e. $|g_i(p_\varepsilon)|\leqslant\rho_M|p_\varepsilon|$), we deduce from the Cauchy–Schwartz inequality and the Poincaré inequality that (2.30) reduces to

$$m_0(\rho_m)^2k_0\int_{\Omega}|\nabla p_\varepsilon|^2\,dx\leqslant C_1\big(1+\|f_P\|^2_{L^2(\Omega)}+\|f_I\|^2_{L^2(\Omega)}+\big\|\rho_1^\star s_1^\star\big\|^2_{L^2(\Omega)}+\big\|\rho_2^\star s_2^\star\big\|^2_{L^2(\Omega)}\big), \tag{2.31}$$

where C_1 depends on h, Ω, k_0, m_0, ρ_m, k_∞.

Take $\phi=-s_{1,\varepsilon}$ in (2.26) and as for (2.20) we establish (2.29). □

Proposition 2.2. *Assume $s_i^\star\geqslant 0$, $\rho_i^\star\geqslant 0$, and $\rho_i^\star s_i^\star$ belongs to $L^2(\Omega)$ $(i=1,2)$. Then, there exists (s_1,p) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$ such that $0\leqslant s_i\leqslant 1$ a.e. in Ω, solution of*

$$\int_{\Omega}\phi\frac{\rho_1(p)s_1-\rho_1^\star s_1^\star}{h}\varphi\,dx+\int_{\Omega}\rho_1(p)M_1(s_1)\mathbf{K}\nabla p\cdot\nabla\varphi\,dx+\int_{\Omega}\mathbf{K}\rho_1(p)\alpha_\eta(s_1)\nabla s_1\cdot\nabla\varphi\,dx$$
$$-\int_{\Omega}\rho_1^2(p)M_1(s_1)\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx+\int_{\Omega}\rho_1(p)s_1f_P\varphi\,dx=\int_{\Omega}\rho_1(p)s_1^If_I\varphi\,dx, \tag{2.32}$$

$$\int_{\Omega}\phi\frac{\rho_2(p)s_2-\rho_2^\star s_2^\star}{h}\xi\,dx+\int_{\Omega}\rho_2(p)M_2(s_2)\mathbf{K}\nabla p\cdot\nabla\xi\,dx+\int_{\Omega}\mathbf{K}\rho_2(p)\alpha_\eta(s_1)\nabla s_2\cdot\nabla\xi\,dx$$
$$-\int_{\Omega}\rho_2^2(p)M_2(s_2)\mathbf{K}\mathbf{g}\cdot\nabla\xi\,dx+\int_{\Omega}\rho_2(p)s_2f_P\xi\,dx=\int_{\Omega}\rho_2(p)s_2^If_I\xi\,dx \tag{2.33}$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.

Proof. From Lemma 2.3, up to a subsequence, we have the following convergences

$$s_{1,\varepsilon} \to s_1 \quad \text{weakly in } H^1_{\Gamma_1}(\Omega),\ \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{2.34}$$

$$p_\varepsilon \to p \quad \text{weakly in } H^1_{\Gamma_1}(\Omega),\ \text{strongly in } L^2(\Omega) \text{ and a.e.in } \Omega. \tag{2.35}$$

As previously, we pass to the limit as ε goes to zero in formulations (2.26)–(2.27) to obtain

$$\int_\Omega \phi \frac{\rho_1(p)Z(s_1) - \rho_1^\star s_1^\star}{h} \varphi\,dx + \int_\Omega \rho_1(p)M_1(s_1)\mathbf{K}\nabla p \cdot \nabla\varphi\,dx + \int_\Omega \mathbf{K}\rho_1(p)\alpha_\eta(s_1)\nabla s_1 \cdot \nabla\varphi\,dx$$
$$- \int_\Omega \rho_1^2(p)M_1(s_1)\mathbf{K}\mathbf{g} \cdot \nabla\varphi\,dx + \int_\Omega \rho_1(p)Z(s_1) f_P \varphi\,dx = \int_\Omega \rho_1(p)s_1^I f_I \varphi\,dx, \tag{2.36}$$

$$\int_\Omega \phi \frac{\rho_2(p)Z(s_2) - \rho_2^\star s_2^\star}{h} \xi\,dx + \int_\Omega \rho_2(p)M_2(s_2)\mathbf{K}\nabla p \cdot \nabla\xi\,dx + \int_\Omega \mathbf{K}\rho_2(p)\alpha_\eta(s_1)\nabla s_2 \cdot \nabla\xi\,dx$$
$$- \int_\Omega \rho_2^2(p)M_2(s_2)\mathbf{K}\mathbf{g} \cdot \nabla\xi\,dx + \int_\Omega \rho_2(p)Z(s_2) f_P \xi\,dx = \int_\Omega \rho_2(p)s_2^I f_I \xi\,dx \tag{2.37}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Let us show $s_i \geqslant 0$ a.e. in Ω. For that, consider $\varphi = -(s_1)^-$, $\xi = -(s_2)^-$ respectively in (2.36) and (2.37). Note that, according to the extension of the mobility of each phase we have $M_i(s_i)(s_i)^- = 0$ and from the definition of the function Z we have also $Z(s_i)(s_i)^- = 0$. One gets

$$\int_\Omega \mathbf{K}\rho_i(p)\alpha_\eta(s_1)\nabla(s_i)^- \cdot \nabla(s_i)^-\,dx = -\frac{1}{h}\int_\Omega \phi\rho_i^\star s_i^\star (s_i)^-\,dx - \int_\Omega \rho_2(p)s_2^I f_I (s_i)^-\,dx. \tag{2.38}$$

The coefficient of capillary diffusion satisfies $\alpha_\eta(s_1) \geqslant \eta$, the left-hand side of (2.38) is negative, then the above equality leads to

$$\eta \int_\Omega \left|\nabla(s_i)^-\right|^2 dx \leqslant 0,$$

which proves the maximum principle since $(s_i)^-$ vanishes on Γ_1. Then we can replace $Z(s_i)$ by s_i in the formulations (2.36)–(2.37) to obtain (2.32)–(2.33). In the same spirit, the functions M_1, M_2 and α which were extended outside $[0, 1]$ operate now only on $[0, 1]$ where they have a physical meaning. $\square$

3. Proof of Theorem 1.3

The proof is based on a semi-discretization method in time [2]. Let be $T > 0$, $N \in \mathbb{N}^*$ and $h = \frac{T}{N}$. We define the following sequence parametrized by h:

$$p_h^0(x) = p^0, \qquad s_{i,h}^0(x) = s_i^0(x) \quad \text{a.e. in } \Omega, \tag{3.1}$$

for all $n \in [0, N-1]$, consider $(s_{1,h}^n, p_h^n) \in L^2(\Omega) \times L^2(\Omega)$ with $0 \leqslant s_{1,h}^n \leqslant 1$, denote by

$(f_P)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_P(\tau)\,d\tau$, $(f_I)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_I(\tau)\,d\tau$ and $(s_i^I)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} s_i^I(\tau)\,d\tau$ for $i = 1, 2$, then define $(s_{1,h}^{n+1}, p_h^{n+1})$ solution of

$$\begin{aligned}
&\phi\frac{\rho_1(p_h^{n+1})s_{1,h}^{n+1} - \rho_1(p_h^n)s_{1,h}^n}{h} - \operatorname{div}\big(\mathbf{K}\rho_1\big(p_h^{n+1}\big)M_1\big(s_{1,h}^{n+1}\big)\nabla p_h^{n+1}\big)\\
&\quad - \operatorname{div}\big(\mathbf{K}\rho_1\big(p_h^{n+1}\big)\alpha_\eta\big(s_{1,h}^{n+1}\big)\nabla s_{1,h}^{n+1}\big) + \operatorname{div}\big(\mathbf{K}\rho_1^2\big(p_h^{n+1}\big)M_1\big(s_{1,h}^{n+1}\big)\mathbf{g}\big)\\
&\quad + \rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}(f_P)_h^{n+1} = \rho_1\big(p_h^{n+1}\big)\big(s_1^I\big)_h^{n+1}(f_I)_h^{n+1}, \qquad (3.2)\\
&\phi\frac{\rho_2(p_h^{n+1})s_{2,h}^{n+1} - \rho_2(p_h^n)s_{2,h}^n}{h} - \operatorname{div}\big(\mathbf{K}\rho_2\big(p_h^{n+1}\big)M_2\big(s_{2,h}^{n+1}\big)\nabla p_h^{n+1}\big)\\
&\quad - \operatorname{div}\big(\mathbf{K}\rho_2\big(p_h^{n+1}\big)\alpha_\eta\big(s_{1,h}^{n+1}\big)\nabla s_{2,h}^{n+1}\big) + \operatorname{div}\big(\mathbf{K}\rho_2^2\big(p_h^{n+1}\big)M_2\big(s_{2,h}^{n+1}\big)\mathbf{g}\big)\\
&\quad + \rho_2\big(p_h^{n+1}\big)s_{1,h}^{n+1}(f_P)_h^{n+1} = \rho_2\big(p_h^{n+1}\big)\big(s_2^I\big)_h^{n+1}(f_I)_h^{n+1}, \qquad (3.3)
\end{aligned}$$

with the boundary condition (1.8). This sequence is well defined for all $n \in [0, N-1]$ by virtue of Proposition 2.2. As a matter of fact, for given $s_{i,h}^n \in [0,1]$ and $\rho_i(p_h^n)s_{i,h}^n \in L^2(\Omega)$, $i = 1, 2$, we construct $(s_{1,h}^{n+1}, p_h^{n+1}) \in H_{\Gamma_1}^1(\Omega) \times H_{\Gamma_1}^1(\Omega)$ so that $s_{1,h}^{n+1} \in [0,1]$.

Lemma 3.1 *(Uniform estimates with respect to h). The solutions of* (3.2)–(3.3) *satisfy*

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \phi\big(\mathcal{H}_1\big(p_h^{n+1}\big)s_{1,h}^{n+1} - \mathcal{H}_1\big(p_h^n\big)s_{1,h}^n\big)\,dx + \frac{1}{h}\int_\Omega \phi\big(\mathcal{H}_2\big(p_h^{n+1}\big)s_{2,h}^{n+1} - \mathcal{H}_2\big(p_h^n\big)s_{2,h}^n\big)\,dx\\
&\quad + \frac{m_0\rho_m^2 k_0}{2}\int_\Omega \big|\nabla p_h^{n+1}\big|^2\,dx \leqslant C\big(1 + \big\|(f_P)_h^{n+1}\big\|_{L^2(\Omega)}^2 + \big\|(f_I)_h^{n+1}\big\|_{L^2(\Omega)}^2\big), \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \phi\big(\big|\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big|^2 - \big|\rho_1\big(p_h^n\big)s_{1,h}^n\big|^2\big)\,dx + k_0\rho_m^2\eta\int_\Omega \big|\nabla s_{1,h}^{n+1}\big|^2\,dx\\
&\quad \leqslant C\big(\big\|(f_P)_h^{n+1}\big\|_{L^2(\Omega)}^2 + \big\|(f_I)_h^{n+1}\big\|_{L^2(\Omega)}^2 + \big\|\nabla p_h^{n+1}\big\|_{L^2(\Omega)}^2\big), \qquad (3.5)
\end{aligned}$$

where C does not depend on h, and the function $\mathcal{H}_i$ is defined in (1.20).

Proof. First of all, let us prove that: for all $s_i \geqslant 0$ and $s_i^\star \geqslant 0$ such that $s_1 + s_2 = s_1^\star + s_2^\star = 1$,

$$\begin{aligned}
&\big(\rho_1(p)s_1 - \rho_1(p^\star)s_1^\star\big)g_1(p) + \big(\rho_2(p)s_2 - \rho_2(p^\star)s_2^\star\big)g_2(p)\\
&\quad \geqslant \mathcal{H}_1(p)s_1 - \mathcal{H}_1(p^\star)s_1^\star + \mathcal{H}_2(p)s_2 - \mathcal{H}_2(p^\star)s_2^\star, \qquad (3.6)
\end{aligned}$$

where

$$g_1(p) = \int_0^p \rho_2(q)\,dq,$$

$$g_2(p) = \int_0^p \rho_1(q)\,dq,$$

$$\mathcal{H}_1(p) = \rho_1(p)g_1(p) - \int_0^p \rho_1(q)\rho_2(q)\,dq,$$

$$\mathcal{H}_2(p) = \rho_2(p)g_2(p) - \int_0^p \rho_1(q)\rho_2(q)\,dq.$$

Let us denote by J the left-hand side of (3.6). We have

$$J = \rho_1(p)g_1(p)s_1 - \rho_1(p^\star)g_1(p)s_1^\star + \rho_2(p)g_2(p)s_2 - \rho_2(p^\star)g_2(p)s_2^\star$$

and from the definition of $\mathcal{H}_i$, one gets

$$J = \mathcal{H}_1(p)s_1 + s_1\int_0^p \rho_1(q)\rho_2(q)\,dq - \rho_1(p^\star)g_1(p)s_1^\star$$
$$+ \mathcal{H}_2(p)s_2 + s_2\int_0^p \rho_1(q)\rho_2(q)\,dq - \rho_2(p^\star)g_2(p)s_2^\star.$$

This expression is equivalent to

$$J = \mathcal{H}_1(p)s_1 - \mathcal{H}_1(p^\star)s_1^\star + s_1\int_0^p \rho_1(q)\rho_2(q)\,dq - \rho_1(p^\star)g_1(p)s_1^\star + \rho_1(p^\star)g_1(p^\star)s_1^\star$$
$$- s_1^\star\int_0^{p^\star} \rho_1(q)\rho_2(q)\,dq + \mathcal{H}_2(p)s_2 - \mathcal{H}_2(p^\star)s_2^\star + s_2\int_0^p \rho_1(q)\rho_2(q)\,dq - \rho_2(p^\star)g_2(p)s_2^\star$$
$$+ \rho_2(p^\star)g_2(p^\star)s_2^\star - s_2^\star\int_0^{p^\star} \rho_1(q)\rho_2(q)\,dq,$$

using now that $s_1 + s_2 = s_1^\star + s_2^\star = 1$, then

$$J = \mathcal{H}_1(p)s_1 - \mathcal{H}_1(p^\star)s_1^\star + \mathcal{H}_2(p)s_2 - \mathcal{H}_2(p^\star)s_2^\star + s_1^\star F_1(p) + s_2^\star F_2(p)$$

with

$$F_1(p) = \int_0^p \rho_1(q)\rho_2(q)\,dq - \rho_1(p^\star)g_1(p) + \rho_1(p^\star)g_1(p^\star) - \int_0^{p^\star} \rho_1(q)\rho_2(q)\,dq,$$

and

$$F_2(p)=\int_0^p \rho_1(q)\rho_2(q)\,dq-\rho_2(p^\star)g_2(p)+\rho_2(p^\star)g_2(p^\star)-\int_0^{p^\star}\rho_1(q)\rho_2(q)\,dq.$$

The assertion (3.6) is satisfied since $F_i(p)\geqslant 0$ for all p, $i=1,2$. Indeed, $F_1(p^\star)=0$, and $F_1'(p)=\rho_2(p)(\rho_1(p)-\rho_1(p^\star))$, then $F_1'(p^\star)=0$, from the monotonicity of ρ_1, we have $F_1'(p)\geqslant 0$, for all $p\geqslant p^\star$ and $F_1'(p)\leqslant 0$, for all $p\leqslant p^\star$, which ensures the positiveness of the function F_1. In the same way, we have $F_2(p)\geqslant 0$, for all $p\in\mathbb{R}$. So, (3.6) is established.

The uniform estimates of Lemma 3.1 are obtained by the same energy estimates as for the proof of Lemma 2.3, being more careful with terms involving $\frac{1}{h}$.

Let us multiply scalarly (3.2) with $g_1(p_h^{n+1})$ and add the scalar product of (3.3) with $g_2(p_h^{n+1})$, we have, by using (3.6),

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \phi\big(\mathcal{H}_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}-\mathcal{H}_1\big(p_h^n\big)s_{1,h}^n+\mathcal{H}_2\big(p_h^{n+1}\big)s_{2,h}^{n+1}-\mathcal{H}_2\big(p_h^n\big)s_{2,h}^n\big)\,dx\\
&\quad+\int_\Omega \rho_1\big(p_h^{n+1}\big)\rho_2\big(p_h^{n+1}\big)M\big(s_{1,h}^{n+1}\big)\mathbf{K}\nabla p_h^{n+1}\cdot\nabla p_h^{n+1}\,dx\\
&\quad-\int_\Omega \rho_1\big(p_h^{n+1}\big)\rho_2\big(p_h^{n+1}\big)\big(\rho_2\big(p_h^{n+1}\big)M_2\big(s_{1,h}^{n+1}\big)+\rho_1\big(p_h^{n+1}\big)M_1\big(s_{1,h}^{n+1}\big)\big)\mathbf{K}\mathbf{g}\cdot\nabla p_h^{n+1}\,dx\\
&\quad+\int_\Omega \big(\rho_1\big(p_h^n\big)s_{1,h}^{n+1}g_1\big(p_h^{n+1}\big)+\rho_2\big(p_h^n\big)s_{2,h}^{n+1}g_2\big(p_h^{n+1}\big)\big)(f_P)_h^{n+1}\,dx\\
&=\int_\Omega \big(\rho_1\big(p_h^n\big)\big(s_1^I\big)_h^{n+1}g_1\big(p_h^{n+1}\big)+\rho_2\big(p_h^n\big)\big(s_2^I\big)_h^{n+1}g_2\big(p_h^{n+1}\big)\big)(f_I)_h^{n+1}\,dx. \qquad (3.7)
\end{aligned}$$

The Cauchy–Schwartz inequality allows to obtain (3.4).

The estimate (3.5) is easily obtained by scalar product of (3.2) with $\rho_1(p_h^{n+1})s_{1,h}^{n+1}$,

$$\begin{aligned}
&\frac{1}{2h}\int_\Omega \phi\big(\big|\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big|^2-\big|\rho_1\big(p_h^n\big)s_{1,h}^n\big|^2\big)\,dx\\
&\quad+\int_\Omega \mathbf{K}\rho_1\big(p_h^{n+1}\big)M_1\big(s_{1,h}^{n+1}\big)\nabla p_h^{n+1}\cdot\nabla\big(\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big)\,dx\\
&\quad+\int_\Omega \mathbf{K}\rho_1\big(p_h^{n+1}\big)\alpha_\eta\big(s_{1,h}^{n+1}\big)\nabla s_{1,h}^{n+1}\cdot\nabla\big(\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big)\,dx\\
&\quad-\int_\Omega \mathbf{K}\rho_1^2\big(p_h^{n+1}\big)M_1\big(s_{1,h}^{n+1}\big)\mathbf{g}\cdot\nabla\big(\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big)\,dx\\
&\quad+\int_\Omega \big|\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big|^2(f_P)_h^{n+1}\,dx\int_\Omega \big|\rho_1\big(p_h^{n+1}\big)\big|^2 s_{1,h}^{n+1}\big(s_1^I\big)_h^{n+1}(f_I)_h^{n+1}\,dx.
\end{aligned}$$

From the third integral, we extract the damping term on the gradient of saturation, the other terms are controlled,

$$\frac{1}{2h}\int_\Omega \phi\big(\big|\rho_1\big(p_h^{n+1}\big)s_{1,h}^{n+1}\big|^2-\big|\rho_1\big(p_h^n\big)s_{1,h}^n\big|^2\big)\,dx+\int_\Omega \mathbf{K}\big|\rho_1\big(p_h^{n+1}\big)\big|^2\alpha_\eta\big(s_{1,h}^{n+1}\big)\nabla s_{1,h}^{n+1}\cdot\nabla s_{1,h}^{n+1}\,dx$$
$$\leqslant C\int_\Omega\big|\nabla p_h^{n+1}\cdot\nabla s_{1,h}^{n+1}\big|\,dx+C\int_\Omega\big|\nabla p_h^{n+1}\big|^2\,dx+C\int_\Omega\big|(f_P)_h^{n+1}\big|+\big|(f_I)_h^{n+1}\big|\,dx.$$

This achieves the proof of Lemma 3.1. □

For a given sequence $(u_h^n)_n$, let us denote

$$u_h(0)=u_h^0,$$
$$u_h(t)=\sum_{n=0}^{N-1}u_h^{n+1}\chi_{]nh,(n+1)h]}(t),\quad \forall t\in\,]0,T], \tag{3.8}$$

and

$$\tilde u_h(t)=\sum_{n=0}^{N-1}\left(\left(1+n-\frac{t}{h}\right)u_h^n+\left(\frac{t}{h}\right)nu_h^{n+1}\right)\chi_{[nh,(n+1)h]}(t),\quad \forall t\in[0,T]. \tag{3.9}$$

Then,

$$\partial_t\tilde u_h(t)=\frac{1}{h}\sum_{n=0}^{N-1}\big(u_h^{n+1}-u_h^n\big)\chi_{]nh,(n+1)h[}(t),\quad \forall t\in[0,T]\setminus\left\{\bigcup_{n=0}^{N}nh\right\}.$$

Let the functions p_h and $s_{i,h}$ be defined as in (3.8). For $i=1,2$, we denote by $r_{i,h}$ the function defined by (3.8) corresponding to $r_{i,h}^n=\rho_i(p_h^n)s_{i,h}^n$ and $\tilde r_{i,h}$ the function defined by (3.9) corresponding to $r_{i,h}^n$. In the same way, we denote by $f_{P,h}$, $f_{I,h}$ and $(s_i^I)_h$ the functions corresponding to $(f_P)_h^{n+1}$, $(f_I)_h^{n+1}$ and $(s_i^I)_h^{n+1}$, respectively.

Proposition 3.1. *The sequence*

$$(p_h)_h \text{ is uniformly bounded in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{3.10}$$
$$(s_{1,h})_h \text{ is uniformly bounded in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{3.11}$$
$$(r_{i,h})_h \text{ is uniformly bounded in } L^2\big(0,T;H^1(\Omega)\big),\quad i=1,2, \tag{3.12}$$
$$(\phi\partial_t\tilde r_{i,h})_h \text{ is uniformly bounded in } L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big),\quad i=1,2. \tag{3.13}$$

Proof. Estimate (3.10) is obtained by multiplying (3.4) by h and summing it from $n=0$ to $n=N-1$,

$$\int_{\Omega} \phi\mathcal{H}_1\big(p_h(T)\big)s_{1,h}(T) + \phi\mathcal{H}_2\big(p_h(T)\big)s_{2,h}(T)\,dx + m_0 p_m^2 k_0 \int_{\Omega} |\nabla p_h|\,dx\,dt$$
$$\leqslant \int_{\Omega} \phi\mathcal{H}_1\big(p_h(0)\big)s_{1,h}(0) + \phi\mathcal{H}_2\big(p_h(0)\big)s_{2,h}(0)\,dx$$
$$+ C\big(1 + \|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)}\big), \tag{3.14}$$

where C is a constant independent of h and η. Then, summing (3.5), we obtain (3.11) by

$$\int_{\Omega} \phi\big|\rho_1\big(p_h(T)\big)s_{1,h}(T)\big|^2\,dx + k_0\rho_m\eta \int_{Q_T} |\nabla s_{1,h}|^2\,dx\,dt$$
$$\leqslant \int_{\Omega} \phi\big|s_{1,h}(0)\big|^2\,dx + C\big(\|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)} + \|\nabla p_h\|^2_{L^2(Q_T)}\big). \tag{3.15}$$

The uniform estimate (3.12) is a consequence of the two previous ones since the densities ρ_i are bounded functions as well as the saturations $0 \leqslant s_{i,h} \leqslant 1$,

$$\nabla r_{i,h} = \sum_{n=0}^{N-1}\big(\rho_i'\big(p_h^{n+1}\big)s_{i,h}^{n+1}\nabla p_h^{n+1} + \rho_i\big(p_h^{n+1}\big)\nabla s_{i,h}^{n+1}\big)\chi_{]nh,(n+1)h]}a(t).$$

From Eqs. (3.2) and (3.3), we have for all $\varphi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$,

$$\langle\phi\partial_t\tilde{r}_{i,h},\varphi\rangle = -\int_{Q_T} \rho_i(p_h)M_i(s_{i,h})\mathbf{K}\nabla p_h\cdot\nabla\varphi\,dx\,dt - \int_{Q_T} \mathbf{K}\rho_i(p_h)\alpha_\eta(s_{1,h})\nabla s_{i,h}\cdot\nabla\varphi\,dx\,dt$$
$$+ \int_{Q_T} \rho_i^2(p_h)M_i(s_{i,h})\mathbf{Kg}\cdot\nabla\varphi\,dx\,dt - \int_{Q_T} \rho_i(p_h)s_{i,h}f_{P,h}\varphi\,dx\,dt$$
$$+ \int_{Q_T} \rho_i(p_h)s_{i,h}^I f_{I,h}\varphi\,dx\,dt.$$

From the above estimates (3.10)–(3.11), (3.13) follows. □

The next step is to pass from an elliptic problem to a parabolic one. Then, we pass to the limit on h, using only compactness on the quantities $r_{i,h}$ $(i = 1,2)$.

Proposition 3.2 *(Convergence with respect to h). We have the following convergences as h goes to zero,*

$$\|r_{i,h} - \tilde{r}_{i,h}\|_{L^2(Q_T)} \to 0, \tag{3.16}$$
$$s_{1,h} \to s_1 \quad \text{weakly in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{3.17}$$
$$p_h \to p \quad \text{weakly in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{3.18}$$
$$r_{i,h} \to r_i \quad \text{strongly in } L^2(Q_T). \tag{3.19}$$

Furthermore,

$$s_{i,h} \to s_i \quad \text{almost everywhere in } Q_T, \tag{3.20}$$

$$0 \leqslant s_i \leqslant 1 \quad \text{almost everywhere in } Q_T, \tag{3.21}$$

$$p_h \to p \quad \text{almost everywhere in } Q_T, \tag{3.22}$$

and

$$r_i = \rho_i(p)s_i \quad \text{almost everywhere in } Q_T. \tag{3.23}$$

Finally, we have

$$\phi\partial_t \tilde{r}_{i,h} \to \phi\partial_t\big(\rho_i(p)s_i\big) \quad \text{weakly in } L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big). \tag{3.24}$$

Proof. Note that

$$\|r_{i,h} - \tilde{r}_{i,h}\|^2_{L^2(Q_T)} = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left\| \left(1+n-\frac{t}{h}\right)\big(r^{n+1}_{i,h} - r^n_{i,h}\big) \right\|^2_{L^2(\Omega)} dt$$
$$= \frac{h}{3} \sum_{n=0}^{N-1} \big\|r^{n+1}_{i,h} - r^n_{i,h}\big\|^2_{L^2(\Omega)}.$$

We multiply scalarly (3.2) and (3.3) respectively with $r^{n+1}_{1,h} - r^n_{1,h}$ and $r^{n+1}_{2,h} - r^n_{2,h}$. Then, summing for $n = 0$ to $N-1$, we get, for $i = 1, 2$,

$$\frac{\phi_0}{h} \sum_{n=0}^{N-1} \big\|r^{n+1}_{i,h} - r^n_{i,h}\big\|^2_{L^2(\Omega)} \leqslant C \sum_{n=0}^{N-1} \big(\big\|\nabla r^n_{i,h}\big\|^2_{L^2(\Omega)} + \big\|\nabla r^{n+1}_{i,h}\big\|^2_{L^2(\Omega)} + \big\|\nabla s^{n+1}_{i,h}\big\|^2_{L^2(\Omega)}$$
$$+ \big\|\nabla p^{n+1}_h\big\|^2_{L^2(\Omega)} + \big\|(f_P)^{n+1}_h\big\|^2_{L^2(\Omega)} + \big\|(f_I)^{n+1}_h\big\|^2_{L^2(\Omega)}\big).$$

This yields to

$$\sum_{n=0}^{N-1} \big\|r^{n+1}_{i,h} - r^n_{i,h}\big\|^2_{L^2(\Omega)} \leqslant C\big(\|\nabla s_{1,h}\|^2_{L^2(Q_T)} + \|\nabla p_h\|^2_{L^2(Q_T)} + \|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)}\big).$$

And from (3.10)–(3.11), we conclude that

$$\|r_{i,h} - \tilde{r}_{i,h}\|_{L^2(Q_T)} \to 0,$$

that is (3.16). From (3.10)–(3.11), the sequences $(p_h)_h$, $(s_{1,h})_h$ are uniformly bounded in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$, we have up to a subsequence the convergence result (3.17)–(3.18).

The sequences $(\tilde{r}_{i,h})_h$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$ thanks to (3.12). In light of (3.13) we have the strong convergence

$$\tilde{r}_{i,h} \to r \quad \text{strongly in } L^2(Q_T).$$

This compactness result is classical and can be found in [4,12] when the porosity is constant, and under the assumption (H1) (the porosity belongs to $W^{1,\infty}(\Omega)$), the proof can be adapted with minor modifications. Then, (3.16) finishes to establish (3.19).

We are now concerned with almost everywhere convergence on pressure p_h and saturations $s_{i,h}$. Recall that

$$s_{1,h} = \frac{r_{1,h}}{\rho_1(p_h)}$$

and

$$r_{2,h} = \rho_2(p_h)s_{2,h} = \rho_2(p_h)\left(1 - \frac{r_{1,h}}{\rho_1(p_h)}\right) = \rho_2(p_h)\left(1 - \frac{r_1}{\rho_1(p_h)}\right) + \frac{\rho_2(p_h)}{\rho_1(p_h)}(r_1 - r_{1,h}).$$

The bounded function ρ_2/ρ_1 and the convergence (3.19) ensure that the following function F of pressure,

$$F(p_h) = \rho_2(p_h)\left(1 - \frac{r_1}{\rho_1(p_h)}\right) \to r_2 \quad \text{almost everywhere in } Q_T. \tag{3.25}$$

Let us show that the function F is increasing. In fact,

$$F'(p_h) = \rho_2'(p_h)\left(1 - \frac{r_1}{\rho_1(p_h)}\right) + \frac{\rho_2(p_h)\rho_1'(p_h)}{\rho_1^2(p_h)}r_1.$$

As $\rho_1(p_h)s_{1,h} \geqslant 0$, the limit r_1 is non-negative.

If $r_1 = 0$, then $F'(p_h) = \rho_2'(p_h) > 0$, else,

$$F'(p_h) = \rho_2'(p_h)(1 - s_{1,h}) + \frac{\rho_2(p_h)\rho_1'(p_h)}{\rho_1^2(p_h)}r_1 + \frac{\rho_2'(p_h)}{\rho_1(p_h)}(r_{1,h} - r_1),$$

and for h small enough,

$$F'(p_h) \geqslant \frac{\rho_2(p_h)\rho_1'(p_h)}{2\rho_1^2(p_h)}r_1 > 0.$$

In any case, the function F is strictly increasing for h small enough, then from (3.25), we have

$$p_h \to F^{-1}(r_2) \quad \text{almost everywhere in } Q_T.$$

Furthermore, (3.18) allows to identify the limit,

$$p_h \to F^{-1}(r_2) = p \quad \text{almost everywhere in } Q_T.$$

At last,

$$s_{i,h} = \frac{r_{i,h}}{\rho_i(p_h)} \to \frac{r_i}{\rho_i(p)} = s_i \quad \text{almost everywhere in } Q_T.$$

The identification of the limit is due to (3.17). This closes the proof. $\square$

The technique for obtaining solutions of non-degenerate system (1.22) is to pass to the limit as h goes to zero on the solutions of

$$\phi\partial_t\tilde r_{i,h} - \operatorname{div}\big(\mathbf{K}\rho_i(p_h)M_i(s_h)\nabla p_h\big) - \operatorname{div}\big(\mathbf{K}\rho_i(p_h)\alpha_\eta(s_{1,h})\nabla s_{i,h}\big) + \operatorname{div}\big(\mathbf{K}\rho_i^2(p_h)M_i(s_h)\mathbf{g}\big) + \rho_i(p_h)s_{i,h}f_{P,h} = \rho(p_h)\big(s_i^I\big)_h f_{I,h}. \tag{3.26}$$

Remark that this system ($i=1,2$) is nothing else than (3.2)–(3.3), written for $n=0$ to $N-1$ by using the definition (3.8) and (3.9). Let us consider the weak formulations ($i=1,2$) on which we have to pass to the limit

$$\begin{aligned}
&\langle\phi\partial_t\tilde r_{i,h},\varphi\rangle + \int_{Q_T}\rho_i(p_h)M_i(s_{i,h})\mathbf{K}\nabla p_h\cdot\nabla\varphi\,dx\,dt \\
&\quad + \int_{Q_T}\mathbf{K}\rho_i(p_h)\alpha_\eta(s_{1,h})\nabla s_{i,h}\cdot\nabla\varphi\,dx\,dt \\
&\quad - \int_{Q_T}\rho_i^2(p_h)M_i(s_{i,h})\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx\,dt + \int_{Q_T}\rho_i(p_h)s_{i,h}f_{P,h}\varphi\,dx\,dt \\
&= \int_{Q_T}\rho_i(p_h)\big(s_i^I\big)_h f_{I,h}\varphi\,dx\,dt,
\end{aligned} \tag{3.27}$$

where φ belongs to $L^2(0,T;H^1_{\Gamma_1}(\Omega))$.

The previous proposition allows us to pass to the limit on each term of (3.27). The first term converges thanks to (3.24), the second one is due to Lebesgue theorem since $\rho_i(p_h)M_i(s_{i,h})\nabla\varphi$ converges almost everywhere in Q_T and is dominated. Then, the strong convergence of this term in $L^2(Q_T)$ and the weak convergence (3.18) establish the convergence of the second term of (3.27) to the desired term.

The third term converges in a similar way, the Lebesgue theorem implies the strong convergence in $(L^2(Q_T))^d$ of $\rho_i(p_h)\alpha_\eta(s_{1,h})\nabla\varphi$ to $\rho_i(p)\alpha_\eta(s_1)\nabla\varphi$, furthermore, from the weak convergence (3.17), the third term converges to the wanted limit. The last two terms converge obviously thanks to the Lebesgue theorem.

We then have established the weak formulation (1.23) of Theorem 1.3. Furthermore, we have well obtained by Proposition 3.2

$$\begin{gathered}
0\leqslant s_1(t,x)\leqslant 1\quad\text{a.e. in } Q_T,\qquad s_1\in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \\
p\in L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big),\qquad \phi\partial_t\big(\rho_i(p)s_i\big)\in L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big),\quad i=1,2.
\end{gathered}$$

The compactness property on $\rho_i(p_h)s_{i,h}$ implies $\rho_i(p)s_i\in C^0(0,T;L^2(\Omega))$, for $i=1,2$. Theorem 1.3 is then proved.

4. Proof of Theorems 1.1 and 1.2

We have shown in Theorem 1.3 that the non-degenerate system (1.22) admits a solution. Here we are going to obtain estimates on the solutions independent of the regularization η. First it is easy to see that the estimates (3.15) is independent of η:

$$\|p^\eta\|_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} \leqslant C, \tag{4.1}$$

where C is a non-negative constant which depends only on $\|f_I\|_{L^2(Q_T)} + \|f_P\|_{L^2(Q_T)}$. The maximum principle on saturation (3.21) is conserved through the limit process,

$$0 \leqslant s_1^\eta(t,x) \leqslant 1 \quad \text{a.e. in } (t,x) \in Q_T. \tag{4.2}$$

We are now concerned with a uniform estimate on the gradient of a degenerate function of the saturation. We investigate separately the cases of the two assumptions (H6) and (H7) to obtain finally the same estimate. Nevertheless, under assumption (H7), additional assumption is required on initial saturation and the injection term $f_I = 0$. The proof is considerably made technical in the case of a double degeneracy (H7).

We state the following two lemmas in order to establish uniform estimates with respect to η.

Lemma 4.1 *(Uniform estimates with respect to η). Assume* (H1)–(H5) *and* (H6) *hold. The solutions to the non-degenerate system* (1.22) *satisfy*:

(i) *The sequence $(\alpha(s_1^\eta)\nabla s_1^\eta)_\eta$ is uniformly bounded in $(L^2(Q_T))^d$.*
(ii) *The sequence $(\eta\nabla s_1^\eta)_\eta$ is uniformly bounded in $(L^2(Q_T))^d$.*
(iii) *The sequence $(\phi(x)\partial_t(\rho_i(p^\eta)s_i^\eta))_\eta$ is uniformly bounded in $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ for $i=1,2$.*

Proof. We take $\varphi = (\rho_1(p^\eta)s_1^\eta)^{r_1+1}$ in the weak formulation (1.23) for $i=1$. We obtain

$$\begin{aligned}
&\frac{1}{r_1+2}\int_\Omega \phi\big(\rho_1\big(p^\eta(T,.)\big)s_1^\eta(T,.)\big)^{r_1+2}\,dx \\
&\quad + (r_1+1)\int_{Q_T} \rho_1\big(p^\eta\big)\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1} M_1\big(s_1^\eta\big)\mathbf{K}\nabla p^\eta \cdot \nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx\,dt \\
&\quad + (r_1+1)\int_{Q_T} \mathbf{K}\rho_1\big(p^\eta\big)\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1}\alpha_\eta\big(s_1^\eta\big)\nabla s_1^\eta \cdot \nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx\,dt \\
&\quad + \int_{Q_T} \rho_1\big(p^\eta\big)s_1^\eta f_P\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1+1}\,dx\,dt \\
&\quad - (r_1+1)\int_{Q_T} \rho_1^2\big(p^\eta\big)\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1} M_1\big(s_1^\eta\big)\mathbf{K}\mathbf{g}\cdot\nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx\,dt \\
&= \frac{1}{r_1+2}\int_\Omega \phi\big(\rho_1(p_0)s_1^0\big)^{r_1+2}\,dx \int_{Q_T} \rho_1\big(p^\eta\big)s_1^I f_I\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1+1}\,dx\,dt.
\end{aligned} \tag{4.3}$$

Arguing that $\nabla(\rho_1(p^\eta)s_1^\eta) = \rho_1(p^\eta)\nabla s_1^\eta + s_1^\eta\nabla\rho_1(p^\eta)$, using the bound (4.1), and the bound of the functions ρ_1, M_1, we obtain the estimate

$$\begin{aligned}
&\frac{1}{r_1+2}\int_\Omega \phi\big(\rho_1\big(p^\eta(T,.)\big)s_1^\eta(T,.)\big)^{r_1+2}\,dx \\
&\qquad + (r_1+1)\int_{Q_T} \mathbf{K}\rho_1^2\big(p^\eta\big)\big(\rho_1\big(p^\eta\big)s_1^\eta\big)^{r_1}\alpha_\eta\big(s_1^\eta\big)\nabla s_1^\eta\cdot\nabla s_1^\eta\,dx\,dt \\
&\quad \leqslant C\bigg(1+\int_{Q_T}\big(s_1^\eta\big)^{r_1}\big|\nabla p^\eta\big|\big|\nabla s_1^\eta\big|\,dx\,dt + \int_{Q_T}\big(s_1^\eta\big)^{r_1}|\mathbf{g}|\big|\nabla s_1^\eta\big|\,dx\,dt \\
&\qquad + \int_{Q_T}\big|\nabla p^\eta\big|^2\,dx\,dt + \int_{Q_T}\big(|f_I|+|f_P|\big)\,dx\,dt\bigg).
\end{aligned}$$

By the Cauchy–Schwartz inequality, and from (H6) $a_0 s_1^{r_1} \leqslant \alpha(s_1) \leqslant A_0 s_1^{r_1}$, we control the right-hand side by the dissipative term,

$$\int_{Q_T}\alpha\big(s_1^\eta\big)^2\nabla s_1^\eta\cdot\nabla s_1^\eta\,dx\,dt \leqslant C + C\int_{Q_T}\big|\nabla p^\eta\big|^2\,dx\,dt + C\int_{Q_T}\big(|f_I|+|f_P|\big)\,dx\,dt,$$

which establish (i) of Lemma 4.1.

In order to establish (ii), let us compute the $L^2(\Omega)$ scalar product of (1.22) ($i=1$) with $\rho_1(p^\eta)s_1^\eta$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \phi\big|\rho_1\big(p^\eta\big)s_1^\eta\big|^2\,dx + \int_\Omega \mathbf{K}\rho_1\big(p^\eta\big)M_1\big(s_1^\eta\big)\nabla p^\eta\cdot\nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx \\
&\qquad + \eta\int_\Omega \mathbf{K}\rho_1\big(p^\eta\big)\nabla s_1^\eta\cdot\nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx + \int_\Omega \mathbf{K}\rho_1\big(p^\eta\big)\alpha\big(s_1^\eta\big)\nabla s_1^\eta\cdot\nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx \\
&\qquad - \int_\Omega \mathbf{K}\rho_1^2\big(p^\eta\big)M_1\big(s_1^\eta\big)\mathbf{g}\cdot\nabla\big(\rho_1\big(p^\eta\big)s_1^\eta\big)\,dx + \int_\Omega \rho_1\big(p^\eta\big)s_1^\eta f_P\rho_1\big(p^\eta\big)s_1^\eta\,dx \\
&\quad = \int_\Omega \rho_1\big(p^\eta\big)s_1^I f_I\rho_1\big(p^\eta\big)s_1^\eta\,dx.
\end{aligned}$$

By direct estimates, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \phi\big|\rho_1\big(p^\eta\big)s_1^\eta\big|^2\,dx + \eta k_0\rho_m\int_\Omega\big|\nabla s_1^\eta\big|^2\,dx + \int_\Omega \mathbf{K}\rho_1^2\big(p^\eta\big)\alpha\big(s_1^\eta\big)\nabla s_1^\eta\cdot\nabla s_1^\eta\,dx \\
&\quad \leqslant C\int_\Omega\big|\nabla s_1^\eta\big|\big|\nabla p^\eta\big|\,dx + C\int_\Omega\big|\nabla p^\eta\big|^2\,dx + C\int_\Omega\big(\big|\nabla s_1^\eta\big|+\big|\nabla p^\eta\big|\big)\,dx \\
&\qquad + C\int_\Omega|f_P|\,dx + C\int_\Omega|f_I|\,dx.
\end{aligned}$$

The estimate (4.1) and the Cauchy–Schwartz inequality lead to the following estimate,

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega}\phi\left|\rho_1\left(p^\eta\right)s_1^\eta\right|^2dx+\eta k_0\rho_m\int_{\Omega}\left|\nabla s_1^\eta\right|^2dx\\&\quad\leqslant C\left(1+\frac{1}{\eta}\right)\int_{\Omega}\left|\nabla p^\eta\right|^2dx+C\int_{\Omega}|f_P|\,dx+C\int_{\Omega}|f_I|\,dx.\end{aligned}$$

Multiplying by η and integrating in time, we obtain the part (ii) of Lemma 4.1.

For all $\varphi\in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ and from (1.22), we have

$$\begin{aligned}&\left|\left\langle\phi(x)\partial_t\left(\rho_i\left(p^\eta\right)s_i^\eta\right),\varphi\right\rangle\right|\\&\quad\leqslant\left|\int_{Q_T}\mathbf{K}\rho_i\left(p^\eta\right)M_i\left(s_i^\eta\right)\nabla p^\eta\cdot\nabla\varphi\,dx\,dt\right|+\left|\int_{Q_T}\mathbf{K}\rho_i\left(p^\eta\right)\alpha\left(s_1^\eta\right)\nabla s_i^\eta\cdot\nabla\varphi\,dx\,dt\right|\\&\qquad+\left|\eta\int_{Q_T}\mathbf{K}\rho_i\left(p^\eta\right)\nabla s_i^\eta\cdot\nabla\varphi\,dx\,dt\right|+\left|\int_{Q_T}\mathbf{K}\rho_i^2\left(p^\eta\right)M_i\left(s_i^\eta\right)\mathbf{g}\cdot\nabla\varphi\,dx\,dt\right|\\&\qquad+\left|\int_{Q_T}\rho_i\left(p^\eta\right)s_i^\eta f_P\varphi\,dx\,dt\right|.\end{aligned}$$

By using (i) and (ii) of Lemma 4.1, we obtain the part (iii) of Lemma 4.1,

$$\begin{aligned}&\left|\left\langle\phi(x)\partial_t\left(\rho_i\left(p^\eta\right)s_i^\eta\right),\varphi\right\rangle\right|\\&\quad\leqslant C\left(1+\left\|\nabla p^\eta\right\|_{L^2(Q_T)}+\left\|\alpha\nabla s_i^\eta\right\|_{L^2(Q_T)}+\left\|\eta\nabla s_i^\eta\right\|_{L^2(Q_T)}\right)\|\nabla\varphi\|_{L^2(Q_T)}\\&\qquad+C\|f_P\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}+C\|f_I\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)}.\qquad\square\end{aligned}$$

Now we are concerned with the same uniform estimates in the case of assumption (H7). For that, let us define the function $\mu(s)\in C^0[0,1[$, that is

$$\mu(s)=\begin{cases}s^{r+1}&\text{for }0\leqslant s\leqslant\xi^\star,\\c^\star\frac{s}{1-s}&\text{for }\xi^\star\leqslant s<1,\end{cases}\tag{4.4}$$

where $c^\star=(1-\xi^\star)(\xi^\star)^r$ and $r\leqslant r_1$ (r_1 is defined in (H7)). The function μ is chosen to satisfy the equation

$$s(1-s)\mu'(s)-\mu(s)=0\quad\text{for }\xi^\star<s<1.\tag{4.5}$$

We also denote the function G as a primitive of μ,

$$G(s)=\int_0^s\mu(z)\,dz.\tag{4.6}$$

Lemma 4.2 *(Uniform estimates with respect to η). Let $f_I = 0$. Assuming* (H1)–(H5), (H7) *and that $G(s_1^0)$ belongs to $L^1(\Omega)$, the solutions to non-degenerate system* (1.22) *satisfy*:

(i) *The sequences $(\alpha^{\frac{1}{2}}(s_1^\eta)\mu'^{\frac{1}{2}}(s_1^\eta)\nabla s_1^\eta)_\eta$ and $(\sqrt{\eta\mu'(s_\eta)}\nabla s_1^\eta)_\eta$ are uniformly bounded in $L^2(Q_T)$.*
(ii) *The sequence $(G(s_1^\eta))_\eta$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$.*
(iii) *The sequence $(\alpha(s_1^\eta)\nabla s_1^\eta)_\eta$ is uniformly bounded in $(L^2(Q_T))^N$.*
(iv) *The sequence $(\eta\nabla s_1^\eta)_\eta$ is uniformly bounded in $(L^2(Q_T))^N$.*
(v) *The sequence $(\phi(x)\partial_t(\rho_i(p^\eta)s_i^\eta))_\eta$ is uniformly bounded in $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ for $i=1,2$.*

Proof. The proof is based on the saturation equation. For that,

$$\partial_t\big(\rho_1(p^\eta)s_1^\eta\big) = \rho_1(p^\eta)\partial_t s_1^\eta + \rho_1'(p^\eta)s_1^\eta\partial_t p^\eta,$$
$$\partial_t\big(\rho_2(p^\eta)s_2^\eta\big) = -\rho_2(p^\eta)\partial_t s_1^\eta + \rho_2'(p^\eta)s_2^\eta\partial_t p^\eta,$$

and denote

$$D(s_1^\eta,p^\eta) = \rho_1(p^\eta)\rho_2'(p^\eta)s_2^\eta + \rho_2(p^\eta)\rho_1'(p^\eta)s_1^\eta,$$

we have

$$\partial_t s_1^\eta = \frac{\rho_2'(p^\eta)s_2^\eta}{D(s_1^\eta,p^\eta)}\partial_t\big(\rho_1(p^\eta)s_1^\eta\big) - \frac{\rho_1'(p^\eta)s_1^\eta}{D(s_1^\eta,p^\eta)}\partial_t\big(\rho_2(p^\eta)s_2^\eta\big). \tag{4.7}$$

From the system (1.22), the equation of saturation is read to

$$\begin{aligned}
&\phi(x)\partial_t s_1^\eta - \frac{\rho_2'(p^\eta)s_2^\eta}{D(s_1^\eta,p^\eta)}\Big(\operatorname{div}\big(\mathbf{K}\rho_1(p^\eta)M_1(s_1^\eta)\nabla p^\eta\big) + \operatorname{div}\big(\mathbf{K}\rho_1(p^\eta)\alpha_\eta(s_1^\eta)\nabla s_1^\eta\big)\\
&\quad - \operatorname{div}\big(\mathbf{K}\rho_1^2(p^\eta)M_1(s_1^\eta)\mathbf{g}\big) - \rho_1(p^\eta)s_1^\eta f_P\Big) + \frac{\rho_1'(p^\eta)s_1^\eta}{D(s_1^\eta,p^\eta)}\Big(\operatorname{div}\big(\mathbf{K}\rho_2(p^\eta)M_2(s_2^\eta)\nabla p^\eta\big)\\
&\quad + \operatorname{div}\big(\mathbf{K}\rho_2(p^\eta)\alpha_\eta(s_1^\eta)\nabla s_2^\eta\big) - \operatorname{div}\big(\mathbf{K}\rho_2^2(p^\eta)M_2(s_2^\eta)\mathbf{g}\big) - \rho_2(p^\eta)s_2^\eta f_P\Big) = 0.
\end{aligned} \tag{4.8}$$

Multiplying Eq. (4.8) by $\mu(s_1^\eta)$ and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi G(s_1^\eta)\,dx + \int_\Omega \Bigg(\mathbf{K}\rho_1(p^\eta)\alpha_\eta(s_1^\eta)\nabla s_1^\eta\cdot\nabla\bigg(\frac{\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\\
&\quad - \mathbf{K}\rho_2(p^\eta)\alpha_\eta(s_1^\eta)\nabla s_2^\eta\cdot\nabla\bigg(\frac{\rho_1'(p^\eta)s_1^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\Bigg)\,dx\\
&\quad + \int_\Omega \Bigg(\mathbf{K}\rho_1(p^\eta)M_1(s_1^\eta)\nabla p^\eta\cdot\nabla\bigg(\frac{\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)
\end{aligned}$$

$$
\begin{aligned}
&- \mathbf{K}\rho_2\big(p^\eta\big)M_2\big(s_2^\eta\big)\nabla p^\eta\cdot\nabla\bigg(\frac{\rho_1'(p^\eta)s_1^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\bigg)\,dx\\
&- \int_\Omega\bigg(\mathbf{K}\rho_1^2\big(p^\eta\big)M_1\big(s_1^\eta\big)\mathbf{g}\cdot\nabla\bigg(\frac{\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\\
&- \mathbf{K}\rho_2^2\big(p^\eta\big)M_2\big(s_2^\eta\big)\mathbf{g}\cdot\nabla\bigg(\frac{\rho_1'(p^\eta)s_1^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\bigg)\,dx\\
&+ \int_\Omega\frac{1}{D(s_1^\eta,p^\eta)}\big(\rho_1\big(p^\eta\big)\rho_2'\big(p^\eta\big)-\rho_2\big(p^\eta\big)\rho_1'\big(p^\eta\big)\big)s_2^\eta s_1^\eta\mu\big(s_1^\eta\big)f_P\,dx=0.
\end{aligned}
\tag{4.9}
$$

Now, let us reduce the second and the third integral for the above equation. We have

$$
\begin{aligned}
&\rho_1\big(p^\eta\big)\nabla\bigg(\frac{\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)\\
&\quad=\frac{\rho_1(p^\eta)\rho_2'(p^\eta)s_2^\eta\mu'(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla s_1^\eta+\frac{\rho_1(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla\rho_2'\big(p^\eta\big)+\frac{\rho_1(p^\eta)\rho_2'(p^\eta)\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla s_2^\eta\\
&\qquad-\frac{\rho_1(p^\eta)\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D^2(s_1^\eta,p^\eta)}\big(\big(\rho_2\big(p^\eta\big)\rho_1'\big(p^\eta\big)-\rho_1\big(p^\eta\big)\rho_2'\big(p^\eta\big)\big)\nabla s_1^\eta+D_p'\big(s_1^\eta,p^\eta\big)\nabla p^\eta\big),
\end{aligned}
$$

where $D_p'(s_1^\eta,p^\eta)$ denotes the derivative of D with respect to p.

Next, from the definition of D, we have

$$
-\frac{\rho_2'(p^\eta)}{D(s_1^\eta,p^\eta)}-\frac{\rho_2'(p^\eta)s_2^\eta}{D^2(s_1^\eta,p^\eta)}\big(\rho_2\big(p^\eta\big)\rho_1'\big(p^\eta\big)-\rho_1\big(p^\eta\big)\rho_2'\big(p^\eta\big)\big)=-\frac{\rho_2'(p^\eta)\rho_2(p^\eta)\rho_1'(p^\eta)}{D^2(s_1^\eta,p^\eta)}
$$

and then

$$
\begin{aligned}
\rho_1\big(p^\eta\big)\nabla\bigg(\frac{\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)&=\frac{\rho_1(p^\eta)\rho_2'(p^\eta)s_2^\eta\mu'(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla s_1^\eta+\frac{\rho_1(p^\eta)s_2^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla\rho_2'\big(p^\eta\big)\\
&\quad-\frac{\rho_1(p^\eta)\rho_2'(p^\eta)\rho_2(p^\eta)\rho_1'(p^\eta)\mu(s_1^\eta)}{D^2(s_1^\eta,p^\eta)}\nabla s_1^\eta\\
&\quad-\frac{\rho_1(p^\eta)\rho_2'(p^\eta)s_2^\eta\mu(s_1^\eta)}{D^2(s_1^\eta,p^\eta)}D_p'\big(s_1^\eta,p^\eta\big)\nabla p^\eta.
\end{aligned}
\tag{4.10}
$$

In the same way, we have

$$
\begin{aligned}
\rho_2\big(p^\eta\big)\nabla\bigg(\frac{\rho_1'(p^\eta)s_1^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\bigg)&=\frac{\rho_2(p^\eta)\rho_1'(p^\eta)s_1^\eta\mu'(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla s_1^\eta+\frac{\rho_2(p^\eta)s_1^\eta\mu(s_1^\eta)}{D(s_1^\eta,p^\eta)}\nabla\rho_1'\big(p^\eta\big)\\
&\quad+\frac{\rho_1(p^\eta)\rho_2'(p^\eta)\rho_2(p^\eta)\rho_1'(p^\eta)\mu(s_1^\eta)}{D^2(s_1^\eta,p^\eta)}\nabla s_1^\eta\\
&\quad-\frac{\rho_2(p^\eta)\rho_1'(p^\eta)s_1^\eta\mu(s_1^\eta)}{D^2(s_1^\eta,p^\eta)}D_p'\big(s_1^\eta,p^\eta\big)\nabla p^\eta.
\end{aligned}
\tag{4.11}
$$

In order to handle the second integral of (4.9), consider the sum of the above equalities (4.10) and (4.11), we obtain

$$I_2 = \int_\Omega \alpha_\eta(s_1^\eta)\mu'(s_1^\eta)\mathbf{K}\nabla s_1^\eta \cdot \nabla s_1^\eta\,dx - \int_\Omega \frac{\rho_1'(p^\eta)\rho_2'(p^\eta)}{D(s_1^\eta, p^\eta)}\alpha_\eta(s_1^\eta)\mu(s_1^\eta)\mathbf{K}\nabla s_1^\eta \cdot \nabla p^\eta\,dx. \tag{4.12}$$

Now the third term from the equality (4.9) is given as

$$I_3 = \int_\Omega f_1(s_1^\eta, p^\eta)\mathbf{K}\nabla s_1^\eta \cdot \nabla p^\eta\,dx + \int_\Omega f_2(s_1^\eta, p^\eta)\mathbf{K}\nabla p^\eta \cdot \nabla p^\eta\,dx, \tag{4.13}$$

where

$$f_1(s_1^\eta, p^\eta) = \biggl(\frac{\rho_1(p^\eta)\rho_2'(p^\eta)M_1(s_1^\eta)s_2^\eta\mu'(s_1^\eta)}{D(s_1^\eta, p^\eta)} - \frac{\rho_2(p^\eta)\rho_1'(p^\eta)s_1^\eta M_2(s_2^\eta)\mu'(s_1^\eta)}{D(s_1^\eta, p^\eta)} - \mu(s_1^\eta)\frac{\rho_2(p^\eta)\rho_2'(p^\eta)\rho_1(p^\eta)\rho_1'(p^\eta)}{D^2(s_1^\eta, p^\eta)}\bigl(M_1(s_1^\eta) + M_2(s_2^\eta)\bigr)\biggr) \tag{4.14}$$

and

$$f_2(s_1^\eta, p^\eta) = \mu(s_1^\eta)\biggl(\frac{\rho_1(p^\eta)\rho_2'' M_1(s_1^\eta)s_2^\eta}{D(s_1^\eta, p^\eta)} - \frac{\rho_2(p^\eta)\rho_1'' M_2(s_2^\eta)s_1^\eta}{D(s_1^\eta, p^\eta)} - \frac{\rho_1(p^\eta)\rho_2'(p^\eta)M_1(s_1^\eta)s_2^\eta}{D^2(s_1^\eta, p^\eta)}D_p'(s_1^\eta, p^\eta) + \frac{\rho_2(p^\eta)\rho_1'(p^\eta)M_2(s_2^\eta)s_1^\eta}{D^2(s_1^\eta, p^\eta)}D_p'(s_1^\eta, p^\eta)\biggr). \tag{4.15}$$

Let us write an equivalent form of the function f_1, we have

$$f_1(s_1^\eta, p^\eta) = M(s_1^\eta)\frac{(\rho_1(p^\eta)\rho_2'(p^\eta))^2}{D^2(s_1^\eta, p^\eta)}\mu'(s_1^\eta)(s_2^\eta)^2 + M(s_1^\eta)\frac{\rho_2(p^\eta)\rho_2'(p^\eta)\rho_1(p^\eta)\rho_1'(p^\eta)}{D^2(s_1^\eta, p^\eta)}\bigl(s_1^\eta s_2^\eta\mu'(s_1^\eta) - \mu(s_1^\eta)\bigr) - M_2(s_2^\eta)\mu'(s_1^\eta). \tag{4.16}$$

Now, we are looking for the dependence of the functions f_1 and f_2 on the function μ. From (4.14), and using the fact that the functions M_1, M_2 are bounded, $M_1(s_1) \leqslant Cs_1$, and ρ_i and their derivatives are bounded, we have

$$\bigl|f_1(s_1^\eta, p^\eta)\bigr|1_{\{s_1^\eta \leqslant \xi^\star\}} \leqslant c_1\bigl(s_1^\eta\mu'(s_1^\eta) + \mu(s_1^\eta)\bigr)1_{\{s_1^\eta \leqslant \xi^\star\}}.$$

In the region where $\{s_1^\eta \geqslant \xi^\star\}$, the second term on the right-hand side of (4.16) vanishes thanks to the definition of μ in this region. This term has motivated the definition of the function μ in the region $\{s_1^\eta \geqslant \xi^\star\}$. Next, the first term on the right-hand side of (4.16) is bounded since $\mu'(s_1^\eta)(s_2^\eta)^2 \leqslant c^\star$ for $\{s_1^\eta \geqslant \xi^\star\}$, which implies the following estimate

$$\left|f_1\left(s_1^\eta, p^\eta\right)\right| \leqslant c_1\left(s_1^\eta \mu'\left(s_1^\eta\right) + \mu\left(s_1^\eta\right)\right) 1_{\{s_1^\eta \leqslant \xi^\star\}} + \left(c_2 + M_2\left(s_2^\eta\right)\mu'\left(s_1^\eta\right)\right) 1_{\{s_1^\eta \geqslant \xi^\star\}}.$$

From the definition of μ in (4.4), and the fact that $M_2(s_2^\eta) \leqslant C(s_2^\eta)^q$ (q is defined in (H7)), it yields

$$\left|f_1\left(s_1^\eta, p^\eta\right)\right| \leqslant c_1(r+2)\left(s_1^\eta\right)^{r+1} 1_{\{s_1^\eta \leqslant \xi^\star\}} + \left(c_2 + c_3\left(s_2^\eta\right)^{q-2}\right) 1_{\{s_1^\eta \geqslant \xi^\star\}}. \tag{4.17}$$

To estimate the function f_2, using the fact that $M_i(s_i) \leqslant C s_i$, and the boundedness of the densities and their derivatives we have

$$\begin{aligned}
\left|f_2\left(s_1^\eta, p^\eta\right)\right| &\leqslant c_3 s_1^\eta s_2^\eta \mu\left(s_1^\eta\right) \leqslant c_3 s_1^\eta \mu\left(s_1^\eta\right) 1_{\{s_1^\eta \leqslant \xi^\star\}} + c_3 s_2^\eta \mu\left(s_1^\eta\right) 1_{\{s_1^\eta \geqslant \xi^\star\}} \\
&\leqslant c_3\left(s_1^\eta\right)^{r+2} 1_{\{s_1^\eta \leqslant \xi^\star\}} + c_3 c^\star 1_{\{s_1^\eta \geqslant \xi^\star\}} \\
&\leqslant c_4,
\end{aligned} \tag{4.18}$$

since $r+2 \geqslant 0$.

Similarly, the fourth term in the equality (4.9) can be expressed as follows

$$I_4 = \int_\Omega f_3\left(s_1^\eta, p^\eta\right) \mathbf{K}\mathbf{g} \cdot \nabla s_1^\eta \, dx + \int_\Omega f_4\left(s_1^\eta, p^\eta\right) \mathbf{K}\mathbf{g} \cdot \nabla p^\eta \, dx, \tag{4.19}$$

where

$$\left|f_3\left(s_1^\eta, p^\eta\right)\right| \leqslant c_1(r+2)\left(s_1^\eta\right)^{r+1} 1_{\{s_1^\eta \leqslant \xi^\star\}} + \left(c_2 + c_3\left(s_2^\eta\right)^{q-2}\right) 1_{\{s_1^\eta \geqslant \xi^\star\}} \tag{4.20}$$

and

$$\left|f_4\left(s_1^\eta, p^\eta\right)\right| \leqslant c_4. \tag{4.21}$$

The fifth term in the equality (4.9) is equivalent to

$$I_5 = \int_\Omega f_5\left(s_1^\eta, p^\eta\right) f_P \, dx \tag{4.22}$$

with

$$f_5\left(s_1^\eta, p^\eta\right) = \frac{1}{D(s_1^\eta, p^\eta)}\left(\rho_1\left(p^\eta\right)\rho_2'\left(p^\eta\right) - \rho_2\left(p^\eta\right)\rho_1'\left(p^\eta\right)\right) s_2^\eta s_1^\eta \mu\left(s_1^\eta\right).$$

As in (4.18), the function f_3 is estimated as follows

$$\left|f_3\left(s_1^\eta, p^\eta\right)\right| \leqslant c_3 s_1^\eta s_2^\eta \mu\left(s_1^\eta\right) \leqslant c_4. \tag{4.23}$$

From (4.12), (4.13) and (4.19), Eq. (4.9) is equivalent to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi G\big(s_1^\eta\big)\,dx + \int_\Omega \alpha_\eta\big(s_1^\eta\big)\mu'\big(s_1^\eta\big)\mathbf{K}\nabla s_1^\eta\cdot\nabla s_1^\eta\,dx \\
&\quad = \int_\Omega \frac{\rho_1'(p^\eta)\rho_2'(p^\eta)}{D(s_1^\eta,p^\eta)}\alpha_\eta\big(s_1^\eta\big)\mu\big(s_1^\eta\big)\mathbf{K}\nabla s_1^\eta\cdot\nabla p^\eta\,dx \\
&\qquad - \int_\Omega f_1\big(s_1^\eta,p^\eta\big)\mathbf{K}\nabla s_1^\eta\cdot\nabla p^\eta\,dx - \int_\Omega f_2\big(s_1^\eta,p^\eta\big)\mathbf{K}\nabla p^\eta\cdot\nabla p^\eta\,dx \\
&\qquad + \int_\Omega f_3\big(s_1^\eta,p^\eta\big)\mathbf{K}\nabla s_1^\eta\cdot\nabla p^\eta\,dx + \int_\Omega f_4\big(s_1^\eta,p^\eta\big)\mathbf{K}\nabla p^\eta\cdot\nabla p^\eta\,dx \\
&\qquad + \int_\Omega f_5\big(s_1^\eta,p^\eta\big)f_P\,dx.
\end{aligned}
\tag{4.24}
$$

Now, let us split the whole integral appearing in (4.24) as $\int_\Omega = \int_{\{s_1^\eta<\xi^\star\}} + \int_{\{s_1^\eta\geqslant\xi^\star\}}$. This decomposition is made in order to handle the degeneracy of the function α on each region, namely near the region $s_1 = 0$ and the region $s_1 = 1$.

Using the definition of α from (H7), and the estimates (4.17), (4.18) and (4.23), we then obtain the estimate

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi G\big(s_1^\eta\big)\,dx + a_0k_0\int_{\{s_1^\eta<\xi^\star\}}\big(s_1^\eta\big)^{r_1}\mu'\big(s_1^\eta\big)\big|\nabla s_1^\eta\big|^2\,dx + a_0k_0\int_{\{s_1^\eta\geqslant\xi^\star\}}\big(s_2^\eta\big)^{r_2}\mu'\big(s_1^\eta\big)\big|\nabla s_1^\eta\big|^2\,dx \\
&\quad + \eta k_0\int_{\{s_1^\eta<\xi^\star\}}\mu'\big(s_1^\eta\big)\big|\nabla s_1^\eta\big|^2\,dx + \eta k_0\int_{\{s_1^\eta\geqslant\xi^\star\}}\mu'\big(s_1^\eta\big)\big|\nabla s_1^\eta\big|^2\,dx \\
&\leqslant c_0A_0k_\infty\int_{\{s_1^\eta<\xi^\star\}}\big(s_1^\eta\big)^{r_1}\mu\big(s_1^\eta\big)\big|\nabla s_1^\eta\cdot\nabla p^\eta\big|\,dx + c_0A_0k_\infty\int_{\{s_1^\eta\geqslant\xi^\star\}}\big(s_2^\eta\big)^{r_2}\mu\big(s_1^\eta\big)\big|\nabla s_1^\eta\cdot\nabla p^\eta\big|\,dx \\
&\quad + \eta c_0k_\infty\int_{\{s_1^\eta<\xi^\star\}}\mu\big(s_1^\eta\big)\big|\nabla s_1^\eta\cdot\nabla p^\eta\big|\,dx + \eta c_0k_\infty\int_{\{s_1^\eta\geqslant\xi^\star\}}\mu\big(s_1^\eta\big)\big|\nabla s_1^\eta\cdot\nabla p^\eta\big|\,dx \\
&\quad + c_1k_\infty(r+2)\int_{\{s_1^\eta<\xi^\star\}}\big(s_1^\eta\big)^{r+1}\big(\big|\nabla s_1^\eta\cdot\nabla p^\eta\big| + \big|\nabla s_1^\eta\big|\big)\,dx \\
&\quad + c_2k_\infty\int_{\{s_1^\eta\geqslant\xi^\star\}}\big(\big|\nabla s_1^\eta\cdot\nabla p^\eta\big| + \big|\nabla s_1^\eta\big|\big)\,dx
\end{aligned}
$$

$$+ c_4 k_\infty \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{q-2} \left(\left|\nabla s_1^\eta \cdot \nabla p^\eta\right| + \left|\nabla s_1^\eta\right|\right) dx$$

$$+ c_4 k_\infty \int\limits_{\Omega} \left|\nabla p^\eta\right|^2 dx + c_4 \int\limits_{\Omega} |f_P|\, dx. \tag{4.25}$$

Denote by J_i, for $i = 1, 9$ the integrals on the right-hand side of the above inequality respectively. We will estimate each term. These estimates are very classical and use essentially the Cauchy–Schwartz inequality, for more clarity, we summarize it below. We denote by C a generic constant independent of η and δ an arbitrary positive real, which will be chosen later. From the definition of μ in (4.4), we have

$$\begin{aligned} J_1 &= c_0 A_0 k_\infty \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r_1+r+1} \left|\nabla s_1^\eta \cdot \nabla p^\eta\right| dx \\ &\leqslant \delta \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r_1+r} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.26}$$

Since $r_2 \geqslant 0$, we have

$$J_2 \leqslant c_0 A_0 k_\infty \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{r_2-1} \left|\nabla s_1^\eta \cdot \nabla p^\eta\right| dx \leqslant \delta \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{r_2-2} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2.$$

From the definition of μ,

$$J_3 \leqslant \eta c_0 k_\infty \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r+1} \left|\nabla s_1^\eta \cdot \nabla p^\eta\right| dx \leqslant \frac{\eta k_0}{2} \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2.$$

One gets also

$$J_4 \leqslant \eta c_0 k_\infty \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{-1} \left|\nabla s_1^\eta \cdot \nabla p^\eta\right| dx \leqslant \frac{\eta k_0}{2} \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{-2} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2.$$

Since $r \geqslant r_1 - 2$ and for every $\delta > 0$,

$$\begin{aligned} J_5 &\leqslant \delta \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{2r+2} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2 \\ &\leqslant \delta \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r_1+r} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2. \end{aligned}$$

From (H7), we have $r_2 \leqslant 2$ and consequently

$$J_6 \leqslant \delta \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{r_2-2} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2.$$

Since $q \geqslant \frac{r_2}{2} + 1$,

$$\begin{aligned} J_7 &\leqslant \delta \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{2q-4} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2 \\ &\leqslant \delta \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{r_2-2} \left|\nabla s_1^\eta\right|^2 dx + C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2. \end{aligned}$$

From the above estimates, and (4.25), we have

$$\begin{aligned} &\frac{d}{dt} \int\limits_{\Omega} \phi G\left(s_1^\eta\right) dx + \left(a_0 k_0 (r+1) - 4\delta\right) \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r_1+r} \left|\nabla s_1^\eta\right|^2 dx \\ &\quad + (a_0 k_0 c^\star - 4\delta) \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_2^\eta\right)^{r_2-2} \left|\nabla s_1^\eta\right|^2 dx + \frac{\eta k_0}{2} \int\limits_{\{s_1^\eta < \xi^\star\}} \left(s_1^\eta\right)^{r} \left|\nabla s_1^\eta\right|^2 dx \\ &\quad + \frac{\eta k_0 c^\star}{2} \int\limits_{\{s_1^\eta \geqslant \xi^\star\}} \left(s_1^\eta\right)^{-2} \left|\nabla s_1^\eta\right|^2 dx \leqslant C \left\|\nabla p^\eta\right\|_{L^2(\Omega)}^2 + C \|f_P\|_{L^1(\Omega)}. \end{aligned} \tag{4.27}$$

Integrating over t between 0 and T, choose δ small enough, and using (4.1), we deduce the parts (ii) and (iii) of this lemma.

Now, let us prove the part (iii) of the lemma. According to (i), it remains to prove that there exists c such that

$$\alpha(s) \leqslant c\mu'(s), \quad \text{for all } s \in [0,1].$$

For $s \leqslant \xi^\star$, as $r \leqslant r_1$, $\alpha(s) \leqslant A_0 s^{r_1} \leqslant A_0 s^r = \frac{A_0}{r+1}\mu'(s)$.

For $s \geqslant \xi^\star$, $\alpha(s) c\mu'(s)$ is obvious since α is bounded and μ' is increasing with $\mu'(\xi^\star) > 0$. Then, for all $s \in [0,1]$, the function α is dominated by the desired function, then (iii) is established.

The second uniform estimate of (i) shows that the sequence $(\sqrt{\eta} 1_{\{s_1^\eta \geqslant \xi^\star\}} \nabla s_1^\eta)_\eta$ is uniformly bounded in $(L^2(Q_T))^N$. Nevertheless (ii) does not allows to bound $(\sqrt{\eta} 1_{\{s_1^\eta \leqslant \xi^\star\}} \nabla s_1^\eta)_\eta$ because of the degeneracy near zero of the function μ', the part (iii) does not follow from (ii). We then reproduce the same computations as in Lemma 4.1 (part (ii)), in order to establish (iv) of Lemma 4.2.

The part (v) is also proved in the same manner as the part (iii) of Lemma 4.1. □

The classical compactness results [10] for evolution equations involve uniform estimate in Sobolev spaces on the solutions and their time derivative. Here, with Lemma 4.1 or Lemma 4.2, we do not have such estimates, as a matter of fact, even if $\partial_t(\rho_i(p^\eta)s_i^\eta)$ is uniformly bounded, we have no estimate on $\nabla(\rho_i(p^\eta)s_i^\eta)$. Also, even if $\nabla\beta(s_1^\eta)$ and ∇p^η are uniformly bounded, we have no estimate on $\partial_t\beta(s_1^\eta)$ and $\partial_t p^\eta$. Nevertheless, note that the couple $(\rho_1(p)s_1, \rho_2(p)s_2)$ is determined in a unique way by the data $(p, \beta(s_1))$. Then, the estimates on $\partial_t(\rho_1(p)s_1, \rho_2(p)s_2)$ and on $\nabla(p, \beta(s_1))$ are sufficient to establish a compactness result on conservative variable $(\rho_1(p)s_1, \rho_2(p)s_2)$.

Lemma 4.3 *(Compactness result for degenerate case). For every M, the set*

$$\begin{aligned}
E_M = \bigl\{&\bigl(\rho_1(p)s_1, \rho_2(p)s_2\bigr) \in L^2(Q_T) \times L^2(Q_T),\ \text{such that } \|p\|_{L^2(0,T;H^1(\Omega))} \leqslant M,\\
&\bigl\|\beta(s_1)\bigr\|_{L^2(0,T;H^1(\Omega))},\ \bigl\|\phi\partial_t\bigl(\rho_1(p)s_1\bigr)\bigr\|_{L^2(0,T;(H^1_{\Gamma_1}(\Omega))')} \leqslant M,\\
&\bigl\|\phi\partial_t\bigl(\rho_2(p)s_2\bigr)\bigr\|_{L^2(0,T;(H^1_{\Gamma_1}(\Omega))')} \leqslant M\bigr\}
\end{aligned}$$

is relatively compact in $L^2(Q_T) \times L^2(Q_T)$, and $\gamma(E_M)$ is relatively compact in $L^2(\Sigma_T) \times L^2(\Sigma_T)$ (γ denotes the trace on Σ_T operator).

Proof. The proof is inspired by the compactness Lemma 3 [4, p. 140], which introduced for incompressible degenerate model. We generalize this result for our compressible degenerate model by a compactness lemma on an implicit set (here E_M).

Denote by

$$u = \rho_1(p)s_1, \qquad v = \rho_2(p)(1 - s_1).$$

Define the map $H : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \times [0, \beta(1)]$ defined by

$$H(u, v) = \bigl(p, \beta(s_1)\bigr), \tag{4.28}$$

where u and v are solutions of the system

$$\begin{aligned}
u\bigl(p, \beta(s_1)\bigr) &= \rho_1(p)\beta^{-1}\bigl(\beta(s_1)\bigr),\\
v\bigl(p, \beta(s_1)\bigr) &= \rho_2(p)\bigl(1 - \beta^{-1}\bigl(\beta(s_1)\bigr)\bigr).
\end{aligned}$$

Note that H is well defined as a diffeomorphism,

$$\begin{aligned}
\begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial \beta} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial \beta} \end{vmatrix} &= -\rho_1'(p)\rho_2(p)\beta^{-1}\bigl(\beta(s_1)\bigr)\bigl(\beta^{-1}\bigr)'\bigl(\beta(s_1)\bigr)\\
&\quad - \rho_1(p)\rho_2'(p)\beta^{-1}\bigl(\beta(s_1)\bigr)\bigl(\beta^{-1}\bigr)'\bigl(\beta(s_1)\bigr) < 0.
\end{aligned}$$

Furthermore, H^{-1} is an Hölder function, it just means that u and v are Hölder functions of order θ with $0 < \theta \leqslant 1$. For that, let (q_1, σ_1) and (q_2, σ_2) in $\mathbb{R}^+ \times [0, \beta(1)]$, we have

$$\begin{aligned}
\bigl|u(q_1, \sigma_1) - u(q_2, \sigma_2)\bigr| &= \bigl|\rho_1(q_1)\beta^{-1}(\sigma_1) - \rho_2(q_2)\beta^{-1}(\sigma_2)\bigr|\\
&\leqslant \bigl|\rho_1(q_1) - \rho_2(q_2)\bigr| + \rho_M\bigl|\beta^{-1}(\sigma_1) - \beta^{-1}(\sigma_2)\bigr|,
\end{aligned}$$

recall that β^{-1} is an Hölder function of order θ, $0 < \theta \leqslant 1$, and using the fact that ρ_1 is bounded, we deduce

$$\begin{aligned}\left|u(q_1,\sigma_1)-u(q_2,\sigma_2)\right| &\leqslant (1+\rho_M)\left|\rho_1(q_1)-\rho_1(q_2)\right|^\theta+\rho_M C_\beta|\sigma_1-\sigma_2|^\theta &\quad (4.29)\\ &\leqslant c_1|q_1-q_2|^\theta+c_2|\sigma_1-\sigma_2|^\theta. &\quad (4.30)\end{aligned}$$

In the same way, we have

$$\left|v(q_1,\sigma_1)-v(q_2,\sigma_2)\right|\leqslant c_1|q_1-q_2|^\theta+c_2|\sigma_1-\sigma_2|^\theta. \quad (4.31)$$

For $0<\tau<1$ and $1<r<\infty$, let us denote

$$W^{\tau,r}(\Omega)=\left\{w\in L^r(\Omega);\int\limits_\Omega\int\limits_\Omega\frac{|w(x)-w(y)|^r}{|x-y|^{d+\tau r}}\,dx\,dy<+\infty\right\}$$

equipped with the norm

$$\|w\|_{W^{\tau,r}(\Omega)}=\left(\|w\|^r_{L^r(\Omega)}+\int\limits_\Omega\int\limits_\Omega\frac{|w(x)-w(y)|^r}{|x-y|^{d+\tau r}}\,dx\,dy\right)^{\frac{1}{r}},$$

recall d denote the space dimension. Let q,σ be in $W^{\tau,r}(\Omega)\times W^{\tau,r}(\Omega)$, then the Hölder functions u and v belong to $W^{\theta\tau,r/\theta}(\Omega)$. In fact, we have

$$\left|u(q,\sigma)\right|\leqslant c_1|q|^\theta+c_2|\sigma|^\theta,$$

then u belongs to $L^{r/\theta}(\Omega)$. Furthermore,

$$\begin{aligned}&\int\limits_\Omega\int\limits_\Omega\frac{|u(q(x),\sigma(x))-u(q(y),\sigma(y))|^{r/\theta}}{|x-y|^{d+\tau r}}\,dx\,dy\\ &\quad\leqslant c_1^{r/\theta}\int\limits_\Omega\int\limits_\Omega\frac{|q(x)-q(y)|^r}{|x-y|^{d+\tau r}}\,dx\,dy+c_2^{r/\theta}\int\limits_\Omega\int\limits_\Omega\frac{|\sigma(x)-\sigma(y)|^r}{|x-y|^{d+\tau r}}\,dx\,dy,\end{aligned}$$

which ensures

$$\left\|u(q,\sigma)\right\|_{W^{\theta\tau,r/\theta}(\Omega)}\leqslant c\left(\|q\|^\theta_{W^{\tau,r}(\Omega)}+\|\sigma\|^\theta_{W^{\tau,r}(\Omega)}\right).$$

Using the continuity of the injection of $H^1(\Omega)$ into $W^{\tau,2}(\Omega)$, with $\tau<1$,

$$\left\|u\big(p,\beta(s_1)\big)\right\|_{W^{\theta\tau,2/\theta}(\Omega)}\leqslant c\left(\|p\|^\theta_{W^{\tau,2}(\Omega)}+\left\|\beta(s_1)\right\|^\theta_{W^{\tau,r}(\Omega)}\right)\leqslant c\left(\|p\|^\theta_{H^1(\Omega)}+\left\|\beta(s_1)\right\|^\theta_{H^1(\Omega)}\right)$$

integrating the above inequality over $(0,T)$,

$$\left\|u\big(p,\beta(s_1)\big)\right\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))}\leqslant c\|p\|^\theta_{L^2(0,T;H^1(\Omega))}+\left\|\beta(s_1)\right\|^\theta_{L^2(0,T;H^1(\Omega))}.$$

Furthermore the porosity function ϕ belongs to $W^{1,\infty}(\Omega)$, it follows that

$$\left\|\phi u\big(p,\beta(s_1)\big)\right\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))}\leqslant C.$$

As Ω is bounded and regular, we have, for $\tau' < \theta\tau$,

$$W^{\theta\tau,2/\theta}(\Omega) \subset W^{\tau',2/\theta}(\Omega) \subset \big(H^1_{\Gamma_1}(\Omega)\big)'$$

with compact injection from $W^{\theta\tau,2/\theta}(\Omega)$ into $W^{\tau',2/\theta}(\Omega)$.

Finally, from a standard compactness argument, we get

$$E_M \text{ is relatively compact in } L^{2/\theta}\big(0,T;W^{\tau',2/\theta}(\Omega)\big) \subset L^2\big(0,T;L^2(\Omega)\big).$$

Secondly, the trace operator γ maps continuously $W^{\tau',2/\theta}(\Omega)$ into $W^{\tau'-\theta/2,2/\theta}(\Gamma)$ as soon as $\tau' > \theta/2$. Choosing for example $\tau' = \frac{3\theta}{4}$, we deduce the relative compactness of $\gamma(E_M)$ into $L^2(\Sigma_T) \times L^2(\Sigma_T)$.

This closes the proof of Lemma 4.3. □

From the previous two lemmas, we deduce the following convergences.

Lemma 4.4 *(Strong and weak convergences). Up to a subsequence the sequences $(s_i^\eta)_\eta$, $(p^\eta)_\eta$ verify the following convergence*

$$p^\eta \to p \qquad \textit{weakly in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{4.32}$$

$$\beta\big(s_1^\eta\big) \to \beta(s_1) \qquad \textit{weakly in } L^2\big(0,T;H^1_{\Gamma_1}(\Omega)\big), \tag{4.33}$$

$$p^\eta \to p \qquad \textit{almost everywhere in } Q_T, \tag{4.34}$$

$$s_1^\eta \to s_1 \qquad \textit{strongly in } L^2(Q_T) \textit{ and } L^2(\Sigma_T), \tag{4.35}$$

$$0 \leqslant s_i(t,x) \leqslant 1 \qquad \textit{almost everywhere in } (t,x) \in Q_T, \tag{4.36}$$

$$\phi\partial_t\big(\rho_i\big(p^\eta\big)s_i^\eta\big) \to \phi\partial_t\big(\rho_i(p)s_i\big) \qquad \textit{weakly in } L^2\big(0,T;\big(H^1_{\Gamma_1}(\Omega)\big)'\big). \tag{4.37}$$

Proof. The first two weak convergences (4.32)–(4.33) follows from the uniform estimates (4.1) and (i) of Lemma 4.1 or (iii) of Lemma 4.2.

Lemma 4.3 ensures the following strong convergences

$$\rho_i\big(p^\eta\big)s_i^\eta \to l_i \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T,$$
$$\rho_i\big(p^\eta\big)s_i^\eta \to l_i \quad \text{in } L^2(\Sigma_T) \text{ and a.e. in } \Sigma_T.$$

As the map H defined in (4.28) is continuous, we deduce

$$p^\eta \to p \qquad \text{a.e. in } Q_T \text{ and a.e. in } \Sigma_T,$$
$$\beta\big(s_1^\eta\big) \to \beta(s_1) \quad \text{a.e. in } Q_T \text{ and a.e. in } \Sigma_T.$$

Remark that the limits have been identified by (4.32) and (4.33).

The convergence (4.34) is then established and as β^{-1} is continuous,

$$s_1^\eta \to s_1 \quad \text{a.e. in } Q_T \text{ and a.e. in } \Sigma_T.$$

From (4.2), the Lebesgue theorem ensures the strong convergence (4.35) and the estimate (4.36) holds.

At last, the weak convergence (4.37) is a consequence of (iii) of Lemma 4.1 or (v) of Lemma 4.2, the identification of the limit follows from the previous convergence. □

In order to achieve the proof of Theorems 1.1 and 1.2, it remains to pass to the limit as η goes to zero in the formulations (1.23), for all smooth test functions $\varphi \in C^1([0,T]; H^1_{\Gamma_1}(\Omega)) \cap L^2(0,T; H^2(\Omega))$ such that $\varphi(T) = 0$

$$\begin{aligned}
&-\int_{Q_T} \phi\rho_i\big(p^\eta\big)s_i^\eta \partial_t\varphi\,dx\,dt + \int_{Q_T} \rho_i\big(p^\eta\big)M_i\big(s_i^\eta\big)\mathbf{K}\nabla p^\eta \cdot \nabla\varphi\,dx\,dt \\
&\quad + \int_{Q_T} \mathbf{K}\rho_i\big(p^\eta\big)\alpha\big(s_1^\eta\big)\nabla s_i^\eta \cdot \nabla\varphi\,dx\,dt + \eta\int_{Q_T} \mathbf{K}\rho_i\big(p^\eta\big)\nabla s_i^\eta \cdot \nabla\varphi\,dx\,dt \\
&\quad - \int_{Q_T} \rho_i^2\big(p^\eta\big)M_i\big(s_i^\eta\big)\mathbf{K}\mathbf{g}\cdot\nabla\varphi\,dx\,dt + \int_{Q_T} \rho_i\big(p^\eta\big)s_i^\eta f_P\varphi\,dx\,dt \\
&= \int_{Q_T} \rho_i\big(p^\eta\big)s_i^I f_I + \int_{\Omega} \phi\rho_i\big(p^0\big)s_i^0\varphi(0,x)\,dx\,\varphi\,dx\,dt, \quad i = 1,2.
\end{aligned} \tag{4.38}$$

The first term converges thanks to the strong convergence of $\rho_i(p^\eta)s_i^\eta$ to $\rho_i(p)s_i$ in $L^2(Q_T)$.

The second term converges arguing in two steps. Firstly, the Lebesgue theorem and the convergences (4.35), (4.34) establish

$$\rho_i\big(p^\eta\big)M_i\big(s_i^\eta\big)\nabla\varphi \to \rho_i(p)M_i(s_i)\nabla\varphi \quad \text{strongly in } \big(L^2(Q_T)\big)^d.$$

Secondly, the weak convergence on pressure (4.32) combined to the above strong convergence validate the convergence for the second term of (4.38).

The third term can be written, for $i = 1$, as

$$\int_{Q_T} \mathbf{K}\rho_1\big(p^\eta\big)\nabla\beta\big(s_1^\eta\big)\cdot\nabla\varphi\,dx\,dt$$

and for $i = 2$, as

$$-\int_{Q_T} \mathbf{K}\rho_2\big(p^\eta\big)\nabla\beta\big(s_1^\eta\big)\cdot\nabla\varphi\,dx\,dt.$$

In any case, the strong convergence of $\rho_i(p^\eta)\nabla\varphi$ in $(L^2(Q_T))^d$ and the weak convergence (4.33) allow to pass to the limit.

Before passing to the limit on the fourth term of (4.38), we first integrate by parts,

$$\eta \int\limits_{Q_T} \rho_i\big(p^\eta\big)\mathbf{K}\nabla s_i^\eta \cdot \nabla\varphi\,dx\,dt$$
$$= -\eta \int\limits_{Q_T} s_i^\eta \operatorname{div}\big(\mathbf{K}^t \rho_i\big(p^\eta\big)\nabla\varphi\big)\,dx\,dt + \eta \int\limits_{\Sigma_T} \rho_i\big(p^\eta\big)s_i^\eta \mathbf{K}^t \nabla\varphi \cdot n\,d\sigma\,dt.$$

The first term of the right-hand side can be upper bounded by

$$C\eta\big(\big\|p^\eta\big\|_{L^2(0,T;H^1_{\Gamma_1}(\Omega))}\|\varphi\|_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} + \|\varphi\|_{L^2(0,T;H^2(\Omega))}\big),$$

which goes trivially to zero thanks to (4.1).

The second term of the right-hand side is dominated by

$$C\eta\big\|s_i^\eta\big\|_{L^2(\Sigma_T)}\|\nabla\varphi\|_{(L^2(\Sigma_T))^d},$$

which goes also to zero by virtue of (4.35).

The last terms converge by using (4.35), (4.34).

The weak formulations (1.13) and (1.14) are then established for smooth test functions. By a density argument, these formulations are also valid for all φ in $L^2(0, T; H^1_{\Gamma_1}(\Omega))$. Theorems 1.1 and 1.2 are then established.

5. Proof of Proposition 1.1

The aim of this section is to show a maximum principle on pressure in the case where the rate of injection and production of fluids are the same, namely $f_I = f_P$. Let us denote by $\overline{\rho_i}(p)$ the continuous extension on $\mathbb{R}$ of the function ρ_i outside $[p_{\min}, p_{\max}]$ (i.e. $\overline{\rho_i}(p) = \rho_i(p_{\max})$ for $p \geqslant p_{\max}$ and $\overline{\rho_i}(p) = \rho_i(p_{\min})$ for $p \leqslant p_{\min}$). Let $p^\star$ be $p_{\max}$ or $p_{\min}$, and denote by $\rho_i^\star = \rho_i(p^\star)$. Note that the system (1.7) can be rewritten in an equivalent form:

$$\begin{aligned}
&\phi\partial_t\big(\big(\rho_1(p) - \rho_1^\star\big)s_1\big) + \phi\rho_1^\star\partial_t s_1 - \operatorname{div}\big(\mathbf{K}\rho_1(p)M_1(s_1)\nabla p\big) - \operatorname{div}\big(\mathbf{K}\big(\overline{\rho_1}(p) - \rho_1^\star\big)\alpha(s_1)\nabla s_1\big)\\
&\quad - \rho_1^\star \operatorname{div}\big(\mathbf{K}\alpha(s_1)\nabla s_1\big) + \big(\overline{\rho_1}(p) - \rho_1^\star\big)s_1 f_P + \rho_1^\star s_1 f_P\\
&= \big(\overline{\rho_1}(p) - \rho_1^\star\big)s_1^I f_I + \rho_1^\star s_1^I f_I,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\phi\partial_t\big(\big(\rho_2(p) - \rho_2^\star\big)s_2\big) + \phi\rho_2^\star\partial_t s_2 - \operatorname{div}\big(\mathbf{K}\rho_2(p)M_2(s_2)\nabla p\big) - \operatorname{div}\big(\mathbf{K}\big(\overline{\rho_2}(p) - \rho_2^\star\big)\alpha(s_1)\nabla s_2\big)\\
&\quad - \rho_2^\star \operatorname{div}\big(\mathbf{K}\alpha(s_1)\nabla s_2\big) + \big(\overline{\rho_2}(p) - \rho_2^\star\big)s_2 f_P + \rho_2^\star s_2 f_P\\
&= \big(\overline{\rho_2}(p) - \rho_2^\star\big)s_2^I f_I + \rho_2^\star s_2^I f_I.
\end{aligned} \tag{5.2}$$

Multiplying Eq. (5.1) by $\rho_2^\star$ and Eq. (5.2) by $\rho_1^\star$ and summing it, we have

$$\begin{aligned}
&\phi\partial_t\big(\rho_2^\star\big(\rho_1(p) - \rho_1^\star\big)s_1\big) + \phi\partial_t\big(\rho_1^\star\big(\rho_2(p) - \rho_2^\star\big)s_2\big) - \operatorname{div}\big(\mathbf{K}\rho_2^\star\rho_1(p)M_1(s_1)\nabla p\big)\\
&\quad - \operatorname{div}\big(\mathbf{K}\rho_1^\star\rho_2(p)M_2(s_2)\nabla p\big) - \operatorname{div}\big(\mathbf{K}\rho_2^\star\big(\overline{\rho_1}(p) - \rho_1^\star\big)\alpha(s_1)\nabla s_1\big)\\
&\quad - \operatorname{div}\big(\mathbf{K}\rho_1^\star\big(\overline{\rho_2}(p) - \rho_2^\star\big)\alpha(s_1)\nabla s_2\big) + \rho_2^\star\big(\overline{\rho_1}(p) - \rho_1^\star\big)s_1 f_P + \rho_1^\star\big(\overline{\rho_2}(p) - \rho_2^\star\big)s_2 f_P\\
&= \rho_2^\star\big(\overline{\rho_1}(p) - \rho_1^\star\big)s_1^I f_I + \rho_1^\star\big(\overline{\rho_2}(p) - \rho_2^\star\big)s_2^I f_I,
\end{aligned} \tag{5.3}$$

since $s_1 + s_2 = s_1^I + s_2^I = 1$ and $f_I = f_P$. Let us denote

$$
\begin{aligned}
Z(s) &= 1 \quad \text{if } s \geqslant 0, \qquad Z(s) = 0 \quad \text{if } s < 0,\\
Y(s) &= s \quad \text{if } s \geqslant 0, \qquad Y(s) = 0 \quad \text{if } s < 0.
\end{aligned}
$$

We first establish that $p(t,.) \leqslant p_{\max}$ and take $p^\star = p_{\max}$.

We formally compute the scalar product of (5.3) with $Z(p-p^\star)$. First, let us remark that

$$
\partial_t\big(Y\big(\rho_2^\star\big(\rho_1(p)-\rho_1^\star\big)\big)s_1\big) = \partial_t\big(\rho_2^\star\big(\rho_1(p)-\rho_1^\star\big)s_1\big)Z(p-p^\star).
$$

Secondly, let us precise the support of the following function,

$$
\overline{\rho_i}(p) - \rho_i^\star = 0 \quad \text{if } p \geqslant p^\star,\ i = 1,2,
$$

so that the following terms of Eq. (5.3)

$$
\begin{aligned}
A = &- \operatorname{div}\big(\mathbf{K}\rho_2^\star\big(\overline{\rho_1}(p)-\rho_1^\star\big)\alpha(s_1)\nabla s_1\big) - \operatorname{div}\big(\mathbf{K}\rho_1^\star\big(\overline{\rho_2}(p)-\rho_2^\star\big)\alpha(s_1)\nabla s_2\big)\\
&+ \rho_2^\star\big(\overline{\rho_1}(p)-\rho_1^\star\big)s_1 f_P + \rho_1^\star\big(\overline{\rho_2}(p)-\rho_2^\star\big)s_2 f_P - \rho_2^\star\big(\overline{\rho_1}(p)-\rho_1^\star\big)s_1^I f_I\\
&+ \rho_1^\star\big(\overline{\rho_2}(p)-\rho_2^\star\big)s_2^I f_I
\end{aligned}
$$

verify

$$
\int_\Omega A Z(p-p^\star) = 0.
$$

Finally, the scalar product of (5.3) with $Z(p-p^\star)$ reduces to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi\big(Y\big(\rho_2^\star\big(\rho_1(p)-\rho_1^\star\big)\big)s_1 + Y\big(\rho_1^\star\big(\rho_2(p)-\rho_2^\star\big)\big)s_2\big)\,dx\\
&\quad + \int_\Omega \mathbf{K}\rho_2^\star\rho_1(p)M_1(s_1)\nabla p\cdot\nabla Z(p-p^\star) + \mathbf{K}\rho_1^\star\rho_2(p)M_2(s_2)\nabla p\cdot\nabla Z(p-p^\star)\,dx = 0.
\end{aligned}
$$

From the positivity of the second integral, we have

$$
\begin{aligned}
&\int_\Omega \phi\big(Y\big(\rho_2^\star\big(\rho_1\big(p(t,.)\big)-\rho_1^\star\big)\big)s_1 + Y\big(\rho_1^\star\big(\rho_2\big(p(t,.)\big)-\rho_2^\star\big)\big)s_2\big)\,dx\\
&\quad \leqslant \int_\Omega \phi\big(Y\big(\rho_2^\star\big(\rho_1(p_0)-\rho_1^\star\big)\big)s_1 + Y\big(\rho_1^\star\big(\rho_2(p_0)-\rho_2^\star\big)\big)s_2\big)\,dx = 0 \quad \text{for all } t \geqslant 0.
\end{aligned}
$$

This establishes formally that

$$
p(t,x) \leqslant p_{\max}, \quad \text{almost everywhere } (t,x) \in \mathbb{R}^+ \times \Omega.
$$

In the same way, considering $p^\star = p_{\min}$, with

$$Z(s) = 1 \quad \text{if } s \leqslant 0, \qquad Z(s) = 0 \quad \text{if } s > 0,$$
$$Y(s) = s \quad \text{if } s \leqslant 0, \qquad Y(s) = 0 \quad \text{if } s > 0,$$

the same computations lead to

$$p(t,x) \geqslant p_{\min}, \quad \text{almost everywhere } (t,x) \in \mathbb{R}^+ \times \Omega.$$

When the function α is not degenerate, we can justify these formal estimates with regularized function approaching Z and H, and give a sense to each term.

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